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Reda El-Dewieny
Assistant Professor, Electrical Power System Engineering Department, Faculty of Engineering, Mansoura University, Mansoura, Egypt.

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APPLICATION OF EIGENVALUE THEORY TO THE ANALYSIS OF TRAVELLING WAVE PROBLEMS IN MULTICONDUCTOR LINES

By

REDA M. K. EL-DEWIENY, B.Sc., Ph.D.

ABSTRACT:

For adequate assessment of transient response in transmission systems, the interest in the problem of travelling waves has grown rapidly over the last two decades. The paper illustrates the important role which the eigenvalue theory plays in the solution of wave problems in multiconductor lines with the principal advantage of setting out the results in a form which is particularly suitable for numerical processing. This contribution also discusses some aspects of the eigenvalue formulation of the multiconductor wave equations and gives a clearer interpretation of the mechanism of propagation.

1. INTRODUCTION:

The solution of travelling wave problems in multiconductor lines has been attempted by some authors[1-3]. Although it would be feasible to solve the equations by an elimination process, this would involve immense practical difficulties in computation. To the author's knowledge, this has not been attempted in practice except in cases where great simplifications have been made to the equations defining the line parameters.

On the assumption that propagation is by means of plane waves only, the variables, when reduced to three only[4], may be separated so that it is necessary to slove a second order partial differential equation defined by the two space variables at right angles to the direction of propagation. Carson[5] formulated a solution for this problem and expressed the results in terms of an impedance and admittance matrix per unit length of the line. There were as many equations as conductors in the system and the main difficulty in obtaining a solution for this set of equations was due to its fast numerical complex.

As digital computers became generally available[6],interest in the solution of multiconductor wave problems was simulated. Adams[6] was the first to interduce the use of matrix algebra in the analysis of asymmetrical systems of conductors. The results were however of limited accuracy because zero conductor resistances were assumed. This assumption was eliminated in other work[1,7], and in 1963 Wedepohl[1] presented a general solution for the problem using the concept of eigenvalues and eigenvector. His approach, however, w lex and a simpler approach using the eigenvalue theory would therefo

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The author is with the Department of Electrical Power Syste
Engineering, Faculty of Engineering, University of El-Mans

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The aim of this paper is to apply the method of eigenvalues and eigenvectors to the solution of wave propagation problems in multiconductor systems using simple matrix manipulation and to clarify further the role which the eigenvalue theory plays in the analysis. It will be shown that, using this theory a simple physical iterpretation of the mechanism of propagation is obtained.

2. BASIC RELATIONSHIPS:

Consider a homogeneous multiconductor line containing n conductors and taking an element of infinitesional length Δx, when current i_j flows in the jth conductor, the voltage developed across the length Δx, of the kth conductor is

$$V_k = - \sum_{j=1}^{n} Z_{kj} \, i_j \, \Delta x$$

where j = 1, 2, 3, ...,n; k goes from 1 to n to include all conductors and Z_{kj} is the mutual impedance per unit length of the kth conductor for current in the jth conductor. It is conventionally assumed that, power flow is from left to right and the reference for measuring distance is at the L.H.S., i.e. the sendind end of the line.

Similarly, there is a shunt displacement current due to the voltage applied to the jth conductor which is

$$i_k = (- Y_{kk} V_k + \sum_{j=1 \neq k}^{n} Y_{kj} V_j) \, \Delta x$$

where Y_{kk} is the self shunt admittance of the kth conductor and Y_{kj} is the mutual shunt admittance between the kth and the jth conductor, both are on a per unit length basis.

Alternatively, for $\Delta x \longrightarrow 0$ the two definitive equations defining an elemental length of the multiconductor system in matrix form are:

$$dV/dx = - ZI \qquad ....(1)$$

$$dI/dx = - YV \qquad ....(2)$$

where Z and Y are the series impedance and shunt admittance matrices (each of order n x n) respectively for the multiconductor line per unit length; V and I are column vectors (order n) defining respectively, the voltage w.r.t. the reference (earth) of, and the current in, any conductor in the system.

The solution of the problem is to find the n values of V and I in terms of specified boundary conditions at system ends. The first step is to eliminate one or the other set of unknowns by differentiating a second time w.r.t. x; viz

$$d^2V/dx^2 = ZY \, V$$

and

$$d^2I/dx^2 = YZ \, I$$

At this stage it is convenient to introduce a matrix S such that

$$S = ZY \qquad \ldots\ldots.(3)$$

Since Z and Y are both bilateral, i.e. $Z_t = Z$ and $Y_t = Y$ giving

$$YZ = Y_t Z_t = (ZY)_t = S_t \qquad \ldots\ldots.(4)$$

so that, in general, $S \neq S_t$ and

$$d^2V/dx^2 = S\ V \qquad \ldots\ldots.(5)$$

$$d^2I/dx^2 = S_t I \qquad \ldots\ldots.(6)$$

3. SOLUTION USING THE EIGENVALUE TECHNIQUE:

Since S is a non-diagonal square matrix of order n x n, it follows that the second order space derivative of each voltage (current) is a function of all voltages (currents). As mentioned in the introduction, it is possible to eliminate (n-1) variables and hence obtain a solution for eqns.(5) and (6). This, however, can be tedious, time consuming and, in addition, the results obtained are neither meaningful nor are they amenable to numerical processing. On the other hand by making use of linear transformation techniques[8], eqns. (5) and (6) will be transformed to diagonal form.

Let $V = P\ v$ $\qquad \ldots\ldots.(7)$

where P is the transformation matrix; non-singular and of order n x n. Substituting in (5) from (7), then

$$d^2v/dx^2 = P^{-1}S\ P\ v = \lambda\ v \qquad \ldots\ldots.(8)$$

where $\lambda = P^{-1}S\ P$ $\qquad \ldots\ldots.(9)$

If P is so chosen that λ is diagonal then the problem reduces to a solution of n simple second order differential equations (8) with the general solution:-

$$v_{x_k} = \exp(-\lambda^{\frac{1}{2}}\ x)\ v_{i_k} + \exp(\lambda^{\frac{1}{2}}\ x)\ v_{r_k}$$

for the voltage w.r.t. earth of conductor k at a distance x along the line. The subscripts i and r refer to 'incident' and 'reflected' waves respectively. The complete matrix solution is:-

$$v = \exp(-\lambda^{\frac{1}{2}}\ x)\ v_i + \exp(\lambda^{\frac{1}{2}}\ x)\ v_r \qquad \ldots\ldots(10)$$

where $\exp(\pm\lambda^{\frac{1}{2}}\ x)$ are diagonal matrices of order n x n, and v, v_i and v_r are column vectors of order n.

Using equation (7), the complete solution in phase quantities is, from (10):-

$$V = P\exp(-\lambda^{\frac{1}{2}}\ x)\ P^{-1}\ V_i + P\exp(\lambda^{\frac{1}{2}}\ x)\ P^{-1}\ V_r \qquad \ldots\ldots(11)$$

where $V_{i,r} = P\ v_{i,r}$.

It is clearly seen that there are 2n unknown constants contained in the vectors of V_i and V_r. These can however be substituted for from the bounary conditions existing on each of the n conductors at each end of the line.

From the foregoing it is evident that λ is the eigenvalue[9] matrix (diagonal of order n x n) of S and P is the corresponding eigenvector matrix (full of order n x n).

Similarly, using eqns.(1) and (11), the solution for the currents through the line conductors is:

$$I = Z^{-1} P \lambda^{\frac{1}{2}} \exp(-\lambda^{\frac{1}{2}} x) P^{-1} V_i - Z^{-1} P \lambda^{\frac{1}{2}} \exp(\lambda^{\frac{1}{2}} x) P^{-1} V_r \quad ......(12)$$

These two results may be simplified if use is made of matrix functions discussed in Appendix 10.1. Hence

$$V = \exp(-\psi x) V_i + \exp(\psi x) V_r \quad ......(13)$$

and by inserting $P^{-1}P$ between '$\lambda^{\frac{1}{2}}$' and 'exp' in both terms of (12), we get

$$I = Y_o \left[\exp(-\psi x) V_i - \exp(\psi x) V_r\right] \quad ......(14)$$

where $Y_o = Z^{-1} \psi = Z^{-1} P \lambda^{\frac{1}{2}} P^{-1}$

The surge or characteristic impedance of a line is that impedance which relates the incident and reflected voltages and currents at any point on the line. This definition implies that, the surge impedance matrix of a multi-conductor line is:

$$Z_o = Y_o^{-1} = \psi^{-1} Z = P \lambda^{-\frac{1}{2}} P^{-1} Z = S^{-\frac{1}{2}} Z \quad ......(15)$$

Y_o is therefore the surge admittance matrix.

It may be seen that matrix equations derived thus far are identical in form to those of a simple transmission line[1]. This is a great advantage resulting from the use of eigenvalue theory.

4. PHYSICAL INTERPRETATION OF THE RESULTS:

A useful insight into the physical meaning of the results may be obtained by considering the case when the incident voltage distribution is proportional to a particular column of P. In this case

$$V_i = a\ P_{(j)}$$

where a is a scalar and $P_{(j)}$ is the jth column of the eigenvector matrix P. Considering a semi-infinte line, in which $V_r = 0$, and applying eqn.(13)

$$V = a \exp(-\psi x) P_{(j)}$$
$$= a\ P \exp(-\lambda^{\frac{1}{2}} x) P^{-1} P_{(j)}$$

Since $P^{-1}P = U$, it follows that $P^{-1}P_{(j)}$ forms column (j) of the unit matrix U and takes the form

$$\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1_{(j)} \\ \vdots \\ 0 \end{bmatrix}$$

that is a zero column with the exception of a unit entry in row j. Hence

$$V = a\ P \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \exp(-\lambda_j^{\frac{1}{2}} x) \\ \vdots \\ 0 \end{bmatrix} = a\ \exp(-\lambda_j^{\frac{1}{2}} x)\ P_{(j)}$$

This important result shows that if the incident voltage distribution is proportional to a particular column of P, then the distribution is preserved; the wave being subjected to a propagation factor with distance. For this reason, columns of P are known as the voltage natural modes, or the eigenvectors of mode distribution vectors of voltage, with associated terms such as modal attenuation, velocity, etc. This important feature is explained in detail by the argument in Appendix 10.2.

5. EIGENVECTORS OF MODE DISTRIBUTION VECTORS OF CURRENT:

Thus far a solution for the wave equations (5) and (6) has been described starting with the differential equation of voltage. The solution so obtained has been formulated in terms of voltages. The sequence may well be reversed and the solution formulated in terms of currents.

Consider eqn.(6) and assuming that

$$I = Q\ i \qquad \text{......(16)}$$

then

$$d^2I/dx^2 = Q^{-1} S_t\ Q\ i = \lambda\ i \qquad \text{......(17)}$$

since the eigenvalues of the transposed matrix are the same as those of the matrix itself(see Appendix 10.3). On the other hand, however, the modal columns of voltage P and those of current Q are not normally the same. In fact, the results of Appendix 10.3 shows that

$$Q = P_t^{-1}$$

Since P^{-1} is always needed for purposes of formulating the matrix functions (Section 3), the eigenvectors of current distribution Q is readily available.

Following the steps of Section 3, the solution in terms of currents is

$$V = Z_o\ [\exp(-\psi_t x)\ I_i - \exp(\psi_t x)\ I_r] \qquad \text{......(18)}$$

$$I = \exp(-\psi_t x)\ I_i + \exp(\psi_t x) \qquad \text{......(19)}$$

where $$Z_o = Y^{-1}\,\psi_t \qquad \text{......(20)}$$

Also by using (14) into (2) we obtain $Y_o = Y\,\psi^{-1}$ and

$$Z_o = \psi\ Y^{-1} \qquad \text{......(21)}$$

which can be seen to be the transpose of (20). It follows that both Z_o and Y_o are symmetric matrices even though S (or ψ) in general is not.

6. EFFECT OF EARTH CONDUCTORS:

So far the solution of the wave equations has been discussed in general terms and it has been shown that there are as many different modes or travelling waves as there are conductors in the system.

As far as the earth conductor is concerned, this can be assumed to be of zero potential at all points along the line, and hence acheiving a reduction in the effective number of modes before a wave solution is attempted. This is true for most practical cases where no resonance effect exists which would cause standing waves to be developed along the earth conductor[10]. Following the procedure explained in Reference 8, the earth wire is eliminated as follows.

The basic Z and Y matrices of the multiconductor line are computed taking into account all conductors including that (those) of earth, and then partitioned into two sub-matrices corresponding to power and earth conductors.

$$\frac{d}{dx}\begin{bmatrix} V_p \\ V_e \end{bmatrix} = -\begin{bmatrix} Z_{pp} & Z_{pe} \\ Z_{ep} & Z_{ee} \end{bmatrix}\begin{bmatrix} I_p \\ I_e \end{bmatrix}$$

and

$$\frac{d}{dx}\begin{bmatrix} I_p \\ I_e \end{bmatrix} = -\begin{bmatrix} Y_{pp} & Y_{pe} \\ Y_{ep} & Y_{ee} \end{bmatrix}\begin{bmatrix} V_p \\ V_e \end{bmatrix}$$

and noting that $V_e = 0$ and $dV_e/dx = 0$, one derives the reduced equations as:-

$$dV_p/dx = -Z'_{pp}\ I_p$$

$$dI_p/dx = -Y'_{pp}\ V_p$$

where $$Z'_{pp} = (Z_{pp} - Z_{pe}\,Z_{ee}^{-1}\,Z_{ep}) \quad \text{and} \quad Y'_{pp} = Y_{pp}$$

and thereafter the solution proceeds as before.

7. EIGENVALUES OF SYSTEMS WITH MIRROR-SYMMETRY CONFIGURATION:

In most double-circuit lines and some single-circuit lines it is possible to choose a plane about which conductors are located in mirror-symmetric pairs. Under these circumstances, it is possible to factorise the matrices (Z, Y and consequently S) in such a way that the order of the factorised matrices is less than that of the original. This results in reducing the order of the polynomial which is solved for the eigenvalues[1] $\lambda_1, \lambda_2, \ldots, \lambda_n$; where n is the number of conductors.

For example, in double-circuit systems the conductors (n = 6) are located in symmetrical pairs about the tower which in this case defines an axis of symmetry. The Z and Y matrices of this system may be partitioned into two sub-matrices with the subscripts a and b denoting the two circuits, and subscripts s and m defining one circuit and the mutual to the other. Thus, applying eqns.(1) and (2)

$$\frac{d}{dx}\begin{bmatrix} V_a \\ V_b \end{bmatrix} = - \begin{bmatrix} Z_s & Z_m \\ Z_m & Z_s \end{bmatrix} \begin{bmatrix} I_a \\ I_b \end{bmatrix}$$

$$\frac{d}{dx}\begin{bmatrix} I_a \\ I_b \end{bmatrix} = - \begin{bmatrix} Y_s & Y_m \\ Y_m & Y_s \end{bmatrix} \begin{bmatrix} V_a \\ V_b \end{bmatrix}$$

where Z_s, Z_m, Y_s and Y_m are submatrices of order $\frac{1}{2}n \times \frac{1}{2}n$; V_a, V_b, I_a and I_b are column vectors of order $\frac{1}{2}n$.

Before applying the P and Q transformation, the following simple transformations are applied; (i.e. $V_p = k \overset{c}{V}$),

$$\begin{bmatrix} V_a \\ V_b \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

and

$$\begin{bmatrix} I_a \\ I_b \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

where 1 is a unit matrix of order $\frac{1}{2}n \times \frac{1}{2}n$ and V_1, V_2, I_1 and I_2 are column vectors of order $\frac{1}{2}n$. These may be called 'primitive components of voltage and currents since they define two primitive circuits which have no mutuals between them. This will be evident below.

The transformed equations become (applying $d\overset{c}{V}/dx = k^{-1} Z_p k \overset{c}{I}$);

$$\frac{d}{dx}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = - \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} Z_s & Z_m \\ Z_m & Z_s \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

$$= - \begin{bmatrix} (Z_s + Z_m) & 0 \\ 0 & (Z_s - Z_m) \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

and similarly for the currents;

$$\frac{d}{dx} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = - \begin{bmatrix} (Y_s + Y_m) & 0 \\ 0 & (Y_s - Y_m) \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

The following reduced equations of order $\frac{1}{2}n$ are thus obtained:

$$\left.\begin{aligned} dV_1/dx &= -(Z_s + Z_m)\ I_1 = -\ Z_1\ I_1 \\ dI_1/dx &= -(Y_s + Y_m)\ V_1 = -\ Y_1\ V_1 \end{aligned}\right] \text{ for the 1st primitive circuit;}$$

and

$$\left.\begin{aligned} dV_2/dx &= -(Z_s - Z_m)\ I_2 = -\ Z_2\ I_2 \\ dI_2/dx &= -(Y_s - Y_m)\ V_2 = -\ Y_2\ V_2 \end{aligned}\right] \text{ for the 2nd primitive circuit.}$$

These equations are solved for the eigenvalues λ_1 for the first set and λ_2 for the second set as if they were separate systems. The pairs of eigenvectors P_1 & Q_1 and P_2 & Q_2 for the separate systems respectively, can then be formed as before.

Then the actual eigenvalues for the complete double-circuit system are:

$$\lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

where λ_1 and λ_2 are diagonal matrices of order $\frac{1}{2}n \times \frac{1}{2}n$, and the actual eigenvectors of the complete system are: (e.g. $P_p = k\ \overset{s}{P}$)

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} = \begin{bmatrix} P_1 & -P_2 \\ P_1 & P_2 \end{bmatrix}$$

and

$$Q = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} = \begin{bmatrix} Q_1 & -Q_2 \\ Q_1 & Q_2 \end{bmatrix}$$

where P_1 , P_2 , Q_1 and Q_2 are all square matrices of order $\frac{1}{2}n \times \frac{1}{2}n$.

Impedances are obtained by an inverse process so that, for example, the characteristic impedance matrix of the actual system is ($Z_p = k\ \overset{s}{Z}\ k^{-1}$);

$$Z_o = \tfrac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} Z_{o1} & 0 \\ 0 & Z_{o2} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \tfrac{1}{2}\begin{bmatrix} (Z_{o1} + Z_{o2}) & (Z_{o1} - Z_{o2}) \\ (Z_{o1} - Z_{o2}) & (Z_{o1} + Z_{o2}) \end{bmatrix}$$

where Z_{o1} and Z_{o2} are the surge impedances of the primitive circuits and each of order $\frac{1}{2}n \times \frac{1}{2}n$. Z_o is of order $n \times n$.

Example:

The above procedure provides by inspection the wave solution for propagation along a telephone line in which the parallel conductors ($n = 2$) are at equal height above ground. Here Z and Y are factorised to give

$$S = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} = \begin{bmatrix} (Z_{11}+Z_{12})(Y_{11}+Y_{12}) & 0 \\ 0 & (Z_{11}-Z_{12})(Y_{11}-Y_{12}) \end{bmatrix}$$

so that, applying $\det(S - \lambda U) = 0$ gives

$$\lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} (Z_{11}+Z_{12})(Y_{11}+Y_{12}) & 0 \\ 0 & (Z_{11}-Z_{12})(Y_{11}-Y_{12}) \end{bmatrix}$$

and the two mode-propagation constants are:

$$\gamma_1 = \lambda_1^{\frac{1}{2}} \quad \text{and} \quad \gamma_2 = \lambda_2^{\frac{1}{2}}$$

Evaluating the matrices of eigenvectors gives

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad Q = P_t^{-1} = \tfrac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

For the surge impedances, applying eqn.(15):

$$Z_{o1} = S_1^{-\frac{1}{2}} Z_1 = \left[(Z_{11}+Z_{12})(Y_{11}+Y_{12})\right]^{-\frac{1}{2}}(Z_{11}+Z_{12})$$

$$= \sqrt{(Z_{11}+Z_{12})/(Y_{11}+Y_{12})}$$

and

$$Z_{o2} = S_2^{-\frac{1}{2}} Z_2 = \left[(Z_{11}-Z_{12})(Y_{11}-Y_{12})\right]^{-\frac{1}{2}}(Z_{11}-Z_{12})$$

$$= \sqrt{(Z_{11}-Z_{12})/(Y_{11}-Y_{12})}$$

Hence

$$Z_o = \frac{1}{2}\begin{bmatrix} A & B \\ B & A \end{bmatrix}$$

where $A = Z_{o1} + Z_{o2}$ and $B = Z_{o1} - Z_{o2}$

It is interesting to note from the structure of P (and Q) that, for the value of λ_1 propagation takes place with the nodal voltages (and currents) equal and in phase in both conductors, and equal and antiphase for the value of λ_2.

8. CONCLUSIONS:

The important role which eigenvalue theory playes in the analysis of multiconductor transmission lines has been demonstrated. The form which the equations take is exactly the same as that of the well known equations of a simple transmission line, provided that the order of multiplication is observed. Furthermore, by using the eigenvalue technique the form of the solution is such that it permits a very simple physical iterpretation of the mechanism of propagation. It is therefore easy to prove that waves travel as linear combinations of surge waves or modes, each of which is associated with a fixed vector distribution of voltage and current, together with an attenuation and velocity factors.

In addition, it has been shown that any incident (or reflected) travelling wave of arbitrary distribution is characterised by a fixed connection between voltage and current vectors known as characteristic (or surge) impedance matrix or its inverse known as the characteristic admittance matrix.

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10. APPENDIXES:

10.1 Matrix functions:

In general, if a matrix S is related to an eigenvalue and eigenvector matrix as follows

$$S = P \lambda P^{-1} = \psi^2, \text{ say}$$

then a function of matrix S, f(S) is defined as

$$f(S) = P\, f(\lambda)\, P^{-1} = f(\psi^2)$$

and since $f(\lambda)$ is diagonal, the matrix function f(S) is defined in terms of known functions. e.g.

$$S^{\frac{1}{2}} = P\, \lambda^{\frac{1}{2}}\, P^{-1} = \psi$$

because $S^{\frac{1}{2}} S^{\frac{1}{2}} = P\, \lambda^{\frac{1}{2}}\, P^{-1}\, P\, \lambda^{\frac{1}{2}}\, P^{-1} = P\, \lambda^{\frac{1}{2}}\, \lambda^{\frac{1}{2}}\, P^{-1} = P\, \lambda\, P^{-1} = S$.

and
$$\begin{aligned} \exp(S) &= U + S + \frac{S^2}{2!} + \frac{S^3}{3!} + \ldots.. \\ &= PUP^{-1} + P\lambda P^{-1} + \frac{P\lambda^2 P^{-1}}{2!} + \frac{P\lambda^3 P^{-1}}{3!} + \ldots\ldots \\ &= P(U + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \ldots..)P^{-1} \\ &= P \exp(\lambda)\, P^{-1} = \exp(\psi^2) \,. \end{aligned}$$

(Note that the exponent of a diagonal matrix is obtained by taking the exponent of each term in the matrix.)

Also, $\exp(\pm S^{\frac{1}{2}} x) = P \exp(\pm \lambda^{\frac{1}{2}} x)\, P^{-1} = \exp(\pm \psi x)$.

10.2 Mode distribution vectors:

It is important to realise that, if the distribution of voltage at the sending end does not correspond to a particular mode, the initial distribution is not necessarily preserved as the wave progresses along the line. Consider a 3-phase horizontal line in which the eigenvector matrix P is given by

$$P = \begin{array}{cc} & \overbrace{\begin{array}{ccc} 1 & 2 & 3 \end{array}}^{\text{modes}} \\ \text{phases}\left\{\begin{array}{c} a \\ b \\ c \end{array}\right. & \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix} \end{array} \quad ; \quad P^{-1} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 0 & -1/2 \\ 1/6 & -1/3 & 1/6 \end{bmatrix}$$

Case 1 :- when the applied voltage at the sending end corresponds to any mode, 1 , say. i.e.

$$V_s = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

Thus, at the sending end

$$v_s = P^{-1} V_s = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

i.e. there is only mode 1 component. The incident voltage on the three conductors respectively are :

mode 1 : 2 2 2
mode 2 : 0 0 0
mode 3 : 0 0 0

and phase voltage : 2 2 2

Since each mode is subjected to its own attenuation factor as it is progressing along the line, the corresponding incident voltage values at the receiving end are (Fig. 1):

mode 1 : 2 b 2 b 2 b
mode 2 : 0 0 0
mode 3 : 0 0 0

and phase voltage : 2 b 2 b 2 b

where b = $\exp(- \lambda_1^{\frac{1}{2}} l)$, l = length of the line, and the units of " $\lambda_1^{\frac{1}{2}} l$" are in nepers.

Case 2 :- when the applied voltage at the sending end does not correspond to any mode, i.e. for example

$$V_s = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Thus, at the sending end

$$v_s = P^{-1} V_s = \begin{bmatrix} 1/3 \\ 0 \\ -1/3 \end{bmatrix}$$

which indicates that there is no mode 2 component. Following the steps of the first case, the incident voltage on the conductors are:

mode 1 : 1/3 1/3 1/3
mode 2 : 0 0 0
mode 3 : -1/3 2/3 -1/3

and phase voltage : 0 1 0

At the receiving end, the corresponding values are (Fig. 2):

mode 1 : (1/3)b (1/3)b (1/3)b
mode 2 : 0 0 0
mode 3 : -(1/3)c (2/3)c -(1/3)c

and phase voltage : $(\frac{1}{3}b - \frac{1}{3}c)$ $(\frac{1}{3}b + \frac{2}{3}c)$ $(\frac{1}{3}b - \frac{1}{3}c)$

where b is as before and c = $\exp(- \lambda_3^{\frac{1}{2}} l)$.

Comparison of the receiving-end phase voltages in both cases shows that, whereas in Case 1 the initial distribution is preserved it is not in Case 2. This leads us to the fact that the structure of the modal columns (determined by the line parameters) is of great importance in some problems. For example where power-line carrier signals are applied to a transmission line, it is desirable to choose a coupling method that results in an incident pattern in which the mode of lowest attenuation is domenant in order to minimise losses.

10.3 Relationship between eigenvalues and eigenvectors of a matrix and its transpose:

Since the eigenvalues of a matrix S are evaluated by solving the polynomial $\det(S - \lambda_{kk} U) = 0$ in λ_{kk}, then the eigenvalues of a transposed matrix must equal those of the original matrix because the determinant of S_t is the same as that of S.

Now if $\quad Q^{-1} S_t\, Q = \lambda$

where Q is the eigenvector matrix of S_t, then by transposition

$$Q_t\, S\, Q_t^{-1} = \lambda_t = \lambda$$

since the transpose of a diagonal matrix is the same as the matrix itself.

But $\quad P^{-1} S\, P = \lambda$ also, then by comparison we have;

$$Q_t = P^{-1} \quad \text{or} \quad Q = P_t^{-1}$$

It can also be shown that $Q = (\Delta P_t)^{-1}$ where Δ is any diagonal matrix.

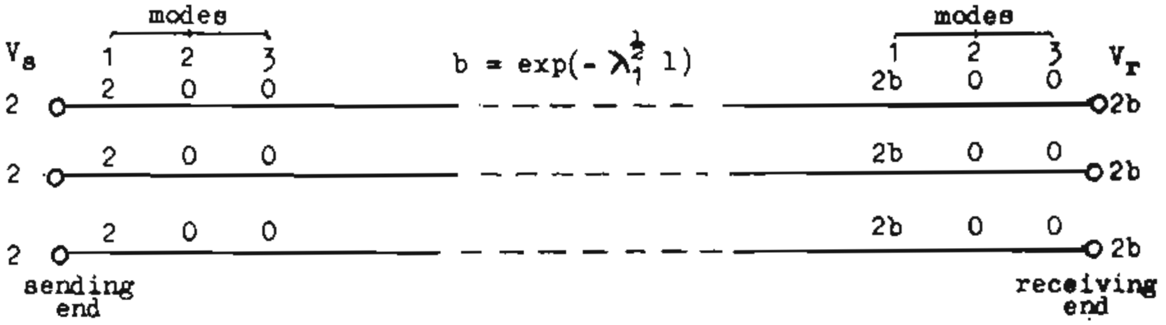

Fig. 1 : Illustration for Case 1 of Appendix 10.2.

(Only incident waves are demonstrated)

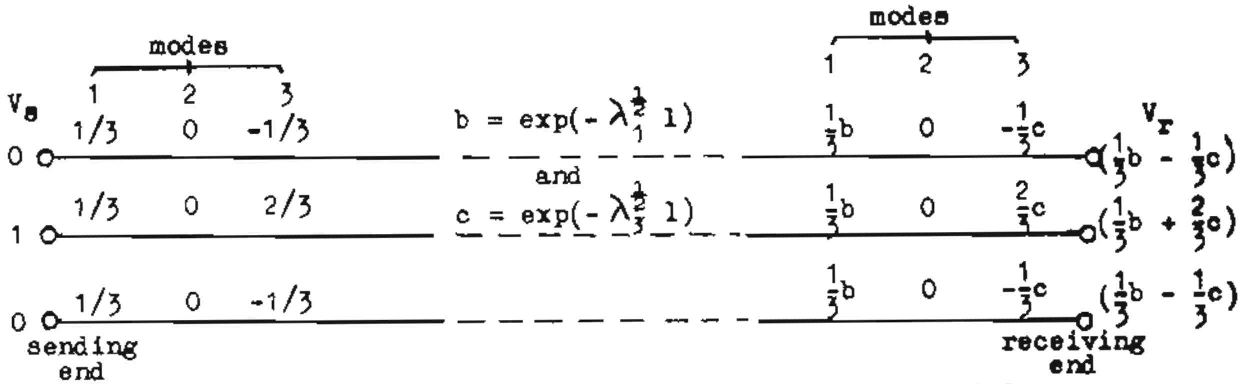

Fig. 2 : Illustration for Case 2 of Appendix 10.2.

(Only incident waves are demonstrated)