Mansoura Engineering Journal

Volume 6 | Issue 1 Article 1

6-1-1981

Reduced Order Modeling Philosophy and Methodology.

E. El-Konyaly
Faculty of Engineering, Mansoura University, Mansoura, Egypt.

A. Montesser
Faculty of Engineering at Menouf, El-Monufia University.

I. El-Dokany
Faculty of Engineering at Menouf, El-Monufia University.

H. Sorror
Faculty of Engineering at Menouf, El-Monufia University.

REDUCED ORDER MODELING PHILOSOPHY AND METHODOLOGY

BY

E.El-Konyaly*, A.Montesser**, I.Eldokany** & H.Sorror**

ABSTRACT:

This work is intended to introduce methods of reduction when the system model is given in state space form. The methods are first introduced, their mathematical foundation is explained in an easy way. Acritical discussion and evaluation of the methods are then presented. A practical network problem is used in the study. Criteria for order determination is then compared to choose the most suitable one for analysis.

1- INTRODUCTION:

The design and control of dynamical systems usually involves the computation of compensating elements and sensitivities due to unmodeled dynamics. This constitutes a computationally expensive problem especially with the increase in size and complexity of modern systems. Reduced order modeling is the historical approach to alleviate the problem of complexity. Simplified equivalent models have been commonly used for analytical studies of linear models.

When the model of the original system is expressed by a set of linear state equations, then the low order models are obtained by approximating the eigenvalues of the system. These models are based on the retention of dominant eigenvalues Ref.(1, 2). The partitioning is dependent on the designer's estimate of the frequency range of the control function. Reduced order models so

* Faculty of Engineering, El-Mansoura University.
** Faculty of Engineering at Menouf, El-Monufia University.

obtained, when excited by the original system input, will usually ignores certain modes of the original system. Erroneous models may be obtained in some even when the original system is very simple and stable.

A ggregation principle Ref (3) is used to improve the situation. This approach can be viewed as a slight generalization of the dominant mode concept. Unfortunately, The aggregated model fail to preserve the structural integrity of the system. In other words, there will be no direct physical interactions between the reduced states. To overcome this difficulty another approach of dynamic approximation, while preserving the structural idendity, is proposed Ref(4,5).

Liapunov functions are, also, used for constructing a reduced order model. This method as first proposed Ref (6) depends heavily on geometry, it is rather restricted to lower order systems. An algebrization view is later appeared in Ref (7), so the method becomes better siuted for modeling higher order systems.

The purpose of the present work is to critically review the previously introduced methods, analyse and compare their characteristics. A numerical example is used for discuasion.

2- <u>System Description</u>:

Consider a linear time invarient system represented by:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \qquad (1)$$

and the partitioned form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \qquad (2)$$

$$y = C_1 x_1 + C_2 x_2 + Du$$

where

x_1 = states to be retained in reduced order model

x_2 = states to be eliminated

A,B,C,D are system's parameters matrices of appropriate dimensions.

3- Dominant Modes Technique :

In this approach, reduced order models can be constructed in either way

1- Truncation

2- Residualization

A truncated reduced order model is obtained on the premise that $x_2 \approx 0$. Thus the reduced order model is described by

$$\begin{aligned} \dot{x}_1 &= A_{11} X_1 + B_1 u \\ y &= C_1 x_1 + D u \end{aligned} \qquad (3)$$

A residualized model attempts to retain the steady state effect of the eliminated states by setting $\dot{x}_2 = 0$ and solving for x_2. This gives

$$x_2 = - A_{22}^{-1} (A_{21} x_1 + B_2 u) \qquad (4)$$

Substitution for x_2 in equation for x_1 results in

$$\begin{aligned} \dot{x}_1 &= (A_{11} - A_{12} A_{22}^{-1} A_{21}) x_1 + (B_1 - A_{12} A_{22}^{-1} B_2)u \\ y_1 &= (C_1 - C_2 A_{22}^{-1} A_{21}) x_1 + (D - C_2 A_{22}^{-1} B_2)u \end{aligned} \tag{5}$$

The partitioning can be intuitively made based on physical reasoning , Otherwise the computation of system eigenvalues have to be calculated and aranged such that $|d_1| < |d_2| < \dots < |d_n|$. Then the retained eigenvalues are chosen on the basis of the frequency range of the control function.

The truncated model suffers from inaccuracies due to the neglect of some of system modes. The residualized model is thus used to improve the steady state response. This can be explained as follows:

A model decomposition is used to bring the system of equation (1) into the form

$$\begin{bmatrix} \dot{\hat{X}}_1 \\ \dot{\hat{X}}_2 \end{bmatrix} = \begin{bmatrix} D_1 & \\ & D_2 \end{bmatrix} \begin{bmatrix} \hat{X}_1 \\ \hat{X}_2 \end{bmatrix} + \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix} u \tag{6}$$

then, the states $\hat{x}_2$ can be approximated as in Ref (8);

$$\hat{x}_2(s) = \lim_{S \to 0} (SI - D_2)^{-1} \hat{B}_2 U(s) \tag{7}$$

and $\hat{x}_2$ can be approximately given by

$$\hat{x}_2(t) \cong - D_2^{-1} \hat{B}_2 u(t) \tag{8}$$

under assumption of non-dominancy of eigenvalues D_2, the D-C transmission between $\hat{x}_2(t)$ and u(t) is affecting the reduced model.

4-Lower Order Aggregated Models:

Because lower order modeling based on dominant mode concept cannot be a satisfactory model in all cases, the developement of an aggregated model becomes apparent.

Consider the system,

$$z = Fz + Gu \tag{9}$$

where F is a lower order coefficient matrix for system of equation (1). In order for (9) to be an aggregated model for (1), we require that

$$z = Qx \tag{10}$$

for all t. This condition is called "dynamic exactness" and is achieved if

$$FQ = QA \quad \text{and} \quad G = QB \tag{11}$$

This means that z is a linear combination of certain of the modes of x. This is quite a restriction for the choice of the class of matrices Q.

The class of matrices Q to fullfil the dynamic exactness property is restricted because the transfer matrix between z and u is

$$H(s) = Q(SI - A)^{-1} B \tag{12}$$

for the original system. For the reduced one is

$$H(s) = (SI - A)^{-1} G \tag{13}$$

since z has lower dimension than x, the equality between (12) and (13) is possible only if there is pole zero cancelation in (12). Thus the matrix Q is restricted to those creating zero to cancel poles of matrix A that are not retained in F.

Two drawbacks associated with aggregation technique are:

1. The computation of an aggregation matrix requires the costly computation of the eigenvectors of A. However under, weak coupling Ref (9) or using efficient algorithms Ref(10), we can operate on small size matrices.

2. The physical interpretation of the state variables of the model is lost. Besides, there are some new interactions which were not present between the original system states. Simply there is no structure preservation.

Another approach which relax the condition of dynamic exactness to one of dynamic approximation while retaining the structural constraints is suggested in Ref (4). As in the dominant modes rechnique, the eigenvalues of the reduced model are chosen from the eigenvalues of the original system. Some elements of the eigenvectors may also be chosen. But, there must be some relaxation on the eigenvectors in order that the specified structure can be realized. Two major problems arises with this technique

1- Non-uniqueness of the solution.
2- Difficult numerical methods required for the computation of the parameter values which will given the desired eigenvalues.

5- Reduction Using Liapunov Functions:

This approach of modeling involves determination of an m-th order model so that the Liapunov surfaces of the model and system have tangential matching. That is, obtain a Liapunov function V_m for the model that has the property

$$\frac{\dot{V}_m}{V_m} \approx \frac{\dot{V}}{V} \qquad (14)$$

in m-dimensional space, where

$$V = X' P X \qquad (15)$$

and

$$\dot{V} = -X' Q X \qquad (16)$$

are Liaponuv function for the orginal system.
Q and P are related by

$$PA + A'P = -Q \qquad (17)$$

and Q is any arbitrary positive definite or semidefinite matrix. Equation (17) has the form

$$P A + \frac{Q}{2} = -(A'P + Q/2) = S \qquad (18)$$

thus, $A = P^{-1}(S - \frac{Q}{2})$ (19)

the reduced order model can be obtained as follows:-

1) Determine the states to be retained.
2) Choose Q to be diagonal with $q_{ii} = 1$, $i \in x_1$ and zero otherwise.
3) Solve (17) for P and (18) for S.
4) Choose P_m from P by deleting those rows and columns belong to x_2. Similarly, we choose S_m.
5) Calculate $A_m = P_m^{-1} (S_m = \frac{Q_m}{2})$. The reduced order model is thus

$$x_m = A_m x_m + B_m u \qquad (20)$$

6) The reduced input matrix B_m is obtained by filting of the original and reduced forced responses at steady state, i.e. $\dot{x}=0$ and $\dot{x}_m = 0$.

$$B_m = \bar{A}_m (C_{11} B_1 + C_{12} B_2) \tag{21}$$

Where

$$A^{-1} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \tag{22}$$

6- Comments and Extensions:

The presented methods share the advantage of giving a reduced order model which can be useful in many control problems. The better understanding the original system, the rigid the model will be. Rigid in the sense of its results.

In obtaining reduced order models, we have two main concerns:

1- Models which successfully represent the original system under variety of operating conditions.
2- The computation involved in obtaining the reduced system should be managable.

However these two requirements are adversely satisfied. Therefore compromise is aften desirable.

From the point of view of accuracy, the aggregated model would be better, however computationally expensive. The dominant mode concept is a rather appealing technique if some criteria is given to decide upon the order of the reduced model. Besides the residualized model can be improved computationally by observing that under similarity transformation

$$x = Mz \tag{23}$$

or

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{24}$$

where M is the model matrix

thus, $x_1 = M_{11} \; z_1 + M_{12} \; z_2$ (25)

and z_1 is the solution of

$$\dot{z}_1 = D_1 \; z_1 + \hat{B}_1 \; u \tag{26}$$

while z_2 can be simply obtained as in equation (8) this yields

$$x_1 = M_{11} \; z_1 + (\hat{B}_1 - M_{12} \; D_2^{-1} \; \hat{B}_2) u \tag{27}$$

Note that the inversion of modal matrix is avoided, which results in computational advantage.

7- Criteria For Model Retention:

The approximate reduced model which closely represents the original system performance can be decided in terms of the largest eigenvalue neglected, the size of the original system and the size of reduced system.

Creteria are obtained for the residualized model by performing a norm bound on the error between the actual rates x_2 and the approximate ones, i.e. by considering $\dot{x}_2 \approx 0$. These creteria are:-

1- With zero initial conditions ans step input, it has been shown Ref(11), that the error is states due to neglecting the higher modes depends apon the relation

E.10. El-Konyaly, Montesser, Eldokany & Sorror

$$u_m = \sqrt{n - m} \Big/ \left| d_{m+1} \right|$$

where n is the order of original system.

m is the order of reduced system

d_{m+1} is the largest eigenvalue to be neglected.

The improvement which can be achieved by increasing the order of the model from m to $m+1$ is measured by

$$V_m = \frac{u_m}{u_{m+1}}$$

2- Relaxing the step input condition and under zero initial condition, the two indices are shown to be Ref(12), to be

$$u_m = (\sqrt{n - m} + 1) / \left| d_{m+1} \right|$$

$$V_m = \frac{u_m}{u_{m+1}}$$

3- In Ref(14) under zero initial condition, it is shown that the orror is bounded to,

$$//e(t)// < k\, u_m$$

where k is a constant and,

$$u_m = (\sqrt{n - m})(\sqrt{n - m + 2}) / \left| d_{m+1} \right|$$

$$V_m = \frac{u_m}{u_{m+1}}$$

4- For uniform bounded input, it is claimed, Ref(15), that under similar conditions as before,

$$u_m = \frac{1}{\left| d_{m+1} \right|} - \frac{1}{R_e\, d_{m+1}}$$

In all creteria, the model of order m is chosen such that V_m will be maximum.

8- **Examples:-**

(1) Consider the 9th order system whose matrix is

$$A=\begin{bmatrix} -2.77 & -3.07 & 2.98 & 0 & 0 & 1 & 0 & 0 & -0.599 \\ 26.8 & -61.5 & 0.524 & 0 & 0 & 0.176 & 0 & -0.0923 & -32.1 \\ 30 & -15.5 & -32.2 & 0 & 0 & 0 & 0 & 0 & -15.6 \\ 0 & 0 & 0 & -27.7 & 0 & 0 & 0 & -0.0828 & 0 \\ 0 & 44 & 0 & -89.8 & -100 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3875 & -100.3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3.33 & 0 & 0 \\ 0 & -223 & 0 & -47.8 & 0 & 0 & 55.4 & -0.35 & -222 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

with input vector $b' = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 3.33 \quad 0 \quad 0]$

The system is reduced to a 3 rd order one in which the retained states are chosen to be X_6, X_8, X_9.

Fig (1), (2) and (3) shows the behaviour of the reduced states when excited by the original input signal for different reduction techniques.

(2) Consider the 15th order system whose eigenvalues are:

$$d_1 = -7.33 \times 10^{-3} \quad , \quad d_2 = -2.09 \times 10^{-1}$$

$$d_{3,4} = -2.77 \times 10^{-1} \pm j\, 3.55 \times 10^{-1}$$

$$d_5 = -3.17 \times 10^{-1}$$

$$d_{6,7} = -3.83 \times 10^{-1} \pm j\, 2.53 \times 10^{-1}$$

$$d_8 = -5.14 \times 10^{-1}$$

$$d_{9,10} = -1.36 \quad \pm\ j\ 3.12$$

$$d_{11,12} = -1.37 \quad \pm\ j\ 4.18$$

$$d_{13,14} = -1.49 \quad \pm\ j\ 3.79 \times 10^{-2}$$

$$d_{15} = -2.4613$$

The methods of mode retention as described in sec (7) are applied to this system and the results are shown in table (1). Based on large V_m we found that creterion of ref (15) is the most powerful one.

(3) Consider the 9th order system with repeated eigenvalues,:

$$-0.2,\ -0.2,\ -0.6 \pm j\ 10\ ,\ -30 \pm j\ 300,\ -40,\ -50,\ -100$$

This example is intended to show that all criteria are failed to provide the correct reduced order except method of ref (15) . Results are given in table (2).

(4) Conclusions:

Different modal reduction techniques are used to obtain reduced order model. It has been concluded that liapunov method can be a good alternative for system reduction if properly programed. Davison's method which is based on retaining dominant eigenvalues is system dependent. The residualized model, however requires good knowledge of the system, is shown to be a good alternative with reduced computation effort.

Table (1) Results of 15th order system

Model Order	Ref(11)		Ref(12)		Ref(14)		Ref(15)	
m	U_m	V_m	U_m	V_m	$\overline{U}_m$	$\overline{V}_m$	U_m	V_m
1	17.903	—	22.6874	—	151.873	—	9.569	—
3	7.695	1.041	9.9158	1.0316	86.6375	1.0779	9.7646	1
5	6.889	1.5187	9.068	1.50165	50.462	1.5826	4.7895	1.31725
7	5.503	1.188	7.448	1.1699	36.631	1.2489	3.891	1.2309
8	0.777	7.079	1.0711	6.9538	11.1686	3.2798	1.40625	2.7669
9	0.720	1.0802	1.0135	1.058	9.6950	1.15199	1.40625	1
10	0.508	1.417	0.7358	1.377	8.2099	1.18089	1.4008	1.0038
11	0.455	1.117	0.682	1.0788	6.7096	1.2236	1.4008	1
12	1.162	0.391	1.833	0.372	5.1879	1.2933	1.3421	1.04379
13	0.948	1.225	1.6198	1.1316	3.6327	1.428	1.3421	1
14	0.406	2.334	0.8125	1.9936	1.2188	2.9805	0.8125	1.6518

Table (2) Results of 9th order system with repeated eigenvalues

Model Order	Ref(11)		Ref(12)		Ref(14)		Ref(15)	
m	U_m	V_m	U_m	V_m	$\overline{U}_m$	$\overline{V}_m$	U_m	V_m
1	14.142	—	14.1421	—	94.1421	—	10	—
2	0.2641	53.5478	0.3639	52.6026	1.66158	56.658	1.9328	5.6625
3	0.2445	1.0801	0.3443	1.0569	1.44235	1.15199	1.9328	1
4	7.4165×10^{-3}	32.96	0.01073	32.087	0.3059	4.7151	0.0583	30.2915
5	6.6335×10^{-3}	1.11803	9.9503×10^{-3}	1.0783	0.25	1.2236	0.0583	1
6	0.0433	0.153198	0.0683	0.14568	0.1933	1.2933	0.05	1.166
7	0.02828	1.5311	0.0482	1.417	0.1082	1.7865	0.04	1.25
8	0.01	2.828	0.02	2.41	0.03	3.6066	0.02	2

R E F E R E N C E S

(1) E.J.Davison " A Method for Simplifiying Linear Dynamic Systems"
IEEETrans. AC-11 PP 93-101, Jan. 1966.

(2) A.Kuppurajulu and S.Elangoven, " simplified power system models for dynamic stability analysis"
IEEETran PAS - 90, PP. 11-23, Jan. 1971.

(3) M.Aoki, "Control of Large Scale Dynamic Systems by Aggregation",
IEEETrans. AC-13 June 1968.

(4) J.E. Van Ness, H. Zimmer and M. Cuttu, "Reduction of dynamic Models of Power systems, 1973 PICA conf. Proc. PP 105-112.

(5) J.E.Van Ness, "Improving Reduced dynamic models power Systems"; 1975 PICA conf. Proc. PP 155-157.

(6) D.M. Nordall and J.L. Melsa, "Modeling with liapunov functions", JACC Preprinte 1967, PP 208-215.

(7) A.K De Sarkar, N. Dharma, " Dynamic System simplification and an Application to Power System stability", Proc IEE vol 119 No 7 July 1972 PP 904-910.

(8) M.R. Chidambara, "Two Simple Techniques for the Simplification of large scale dynamic systems," JACC preprints, 1969, PP 669-674.

(9) R.D. Milne, " The analysis of weakly coupled dynamical systems," Int.J.Cont. vol 2 Aug. 1965.

(10) F.F. Wu and N.Narasimhamurthi," A new Algonthm for Model reduction", Memo ERL-M613 Univ. of calif. Berkeley 1976.

(11) G.B. Mahapatra, " A note on selecting a low order by Davison's Model Simplification technique", IEEE Tran A C-22 NO4 PP 677-678, Aug 1977.

(12) G.B. Mohapatra, " Afurther note on selecting a low order system using the dominant eigenvalue concept," IEEETrans. AC-24 PP 135-136, Feb., 1979.

(13) A.S.Rao, S.S.Lamba and S.V. Rao, "Comments on a note on selecting low order system by davison's simplification technique" IEEETrans. AC -24 PP 141-142 Feb., 1979.

(14) Z.Elrazaz and N.K sinha, " on the selection of the dominant poles of a system to be retained in a low order model", IEEETrans AC-24, PP 792-793 Oct., 1979.

(15) W.H. Enright and M.S. Kamel, " on selecting a low order model using dominant mode concept,", IEEETrans AC-25 PP 976-978, Oct., 1980.

(16) HODA S. SOROUR MSC. Thesis Faculty of Electronic Engineering Monufia University 1981.

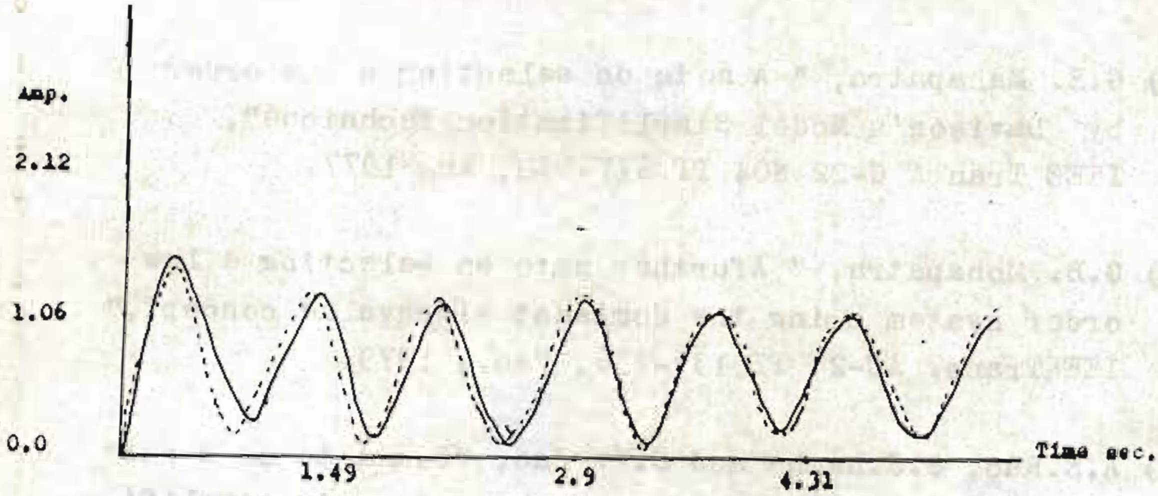

Fig. 1a Time response of state X_1 (Davison's method).

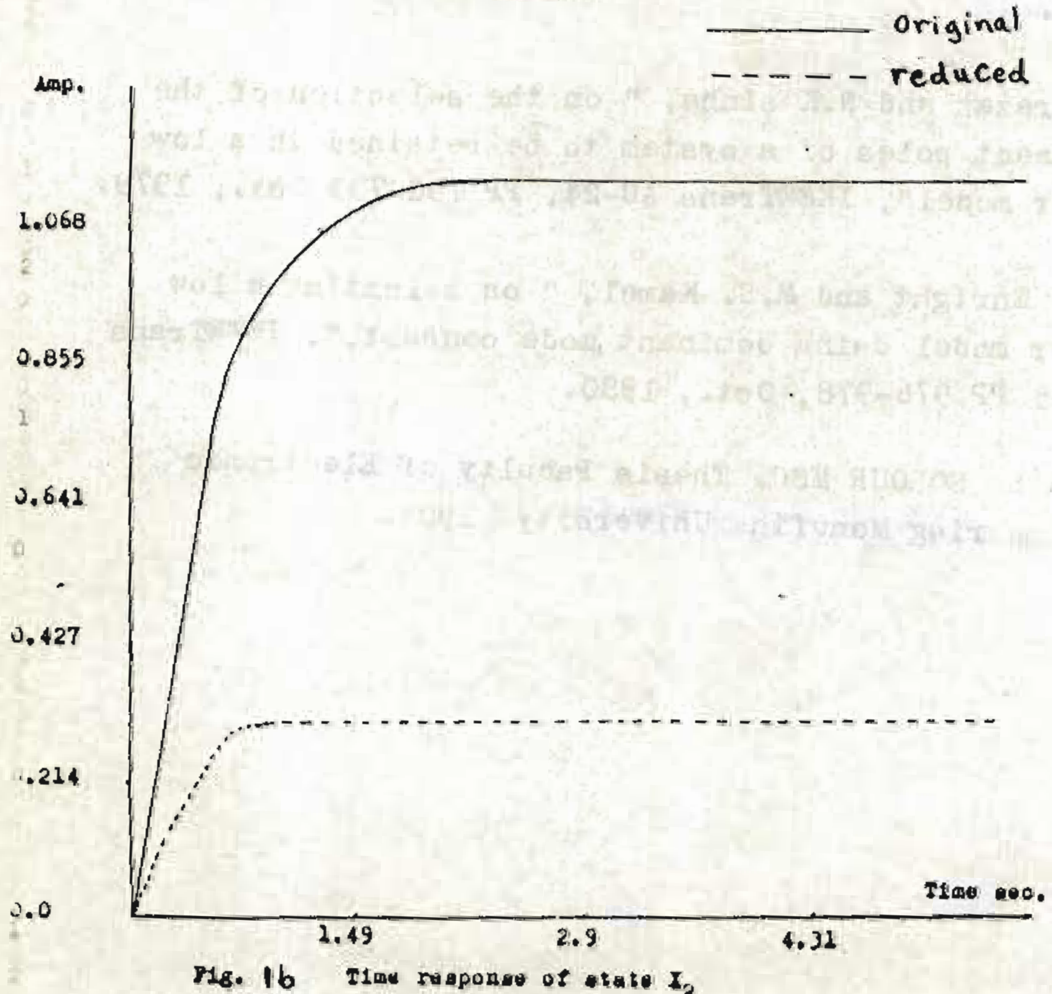

Fig. 1b Time response of state X_2 (Davison's method).

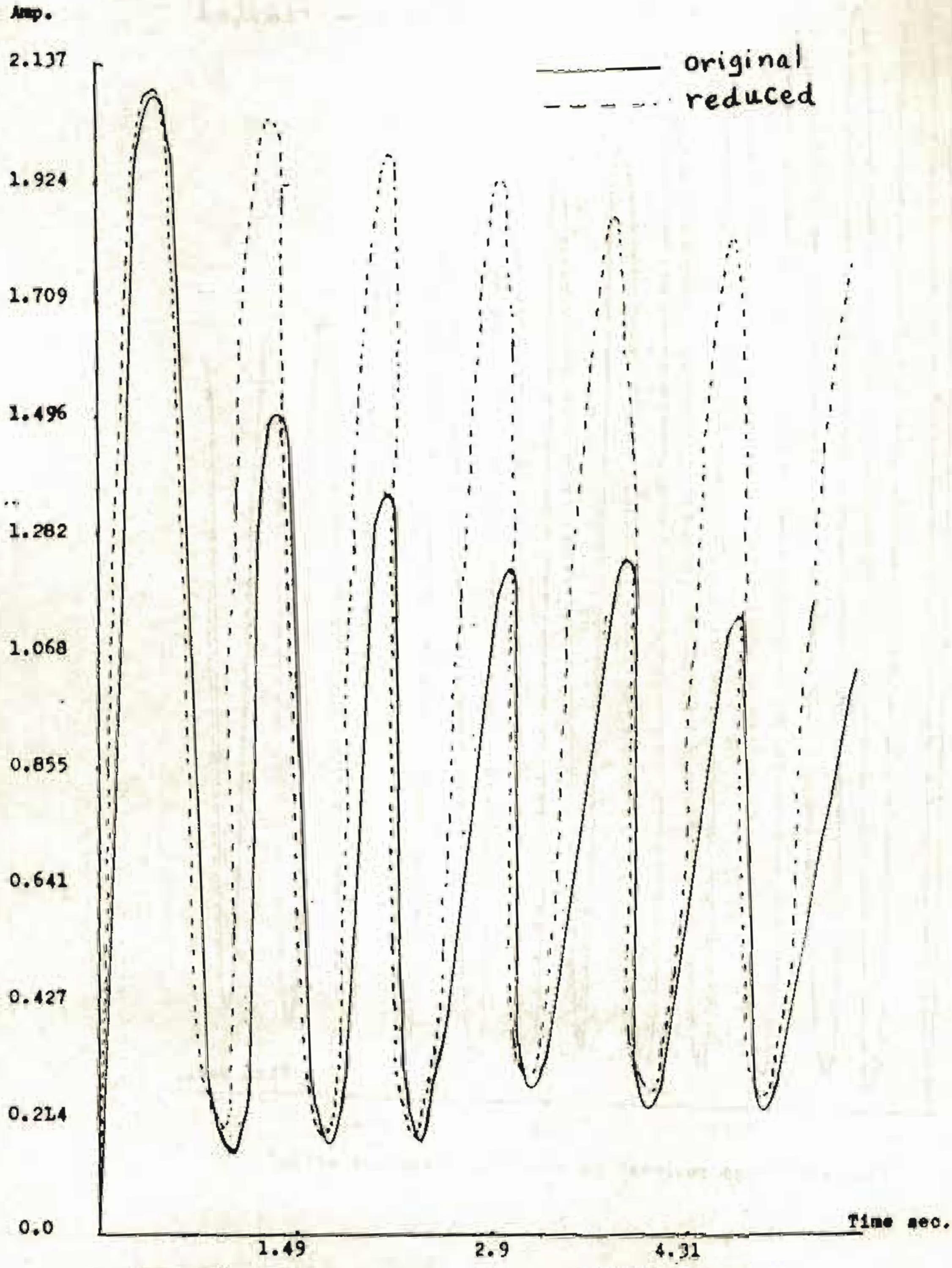

Fig. 1C Time response of state X_3 (Davison's method).

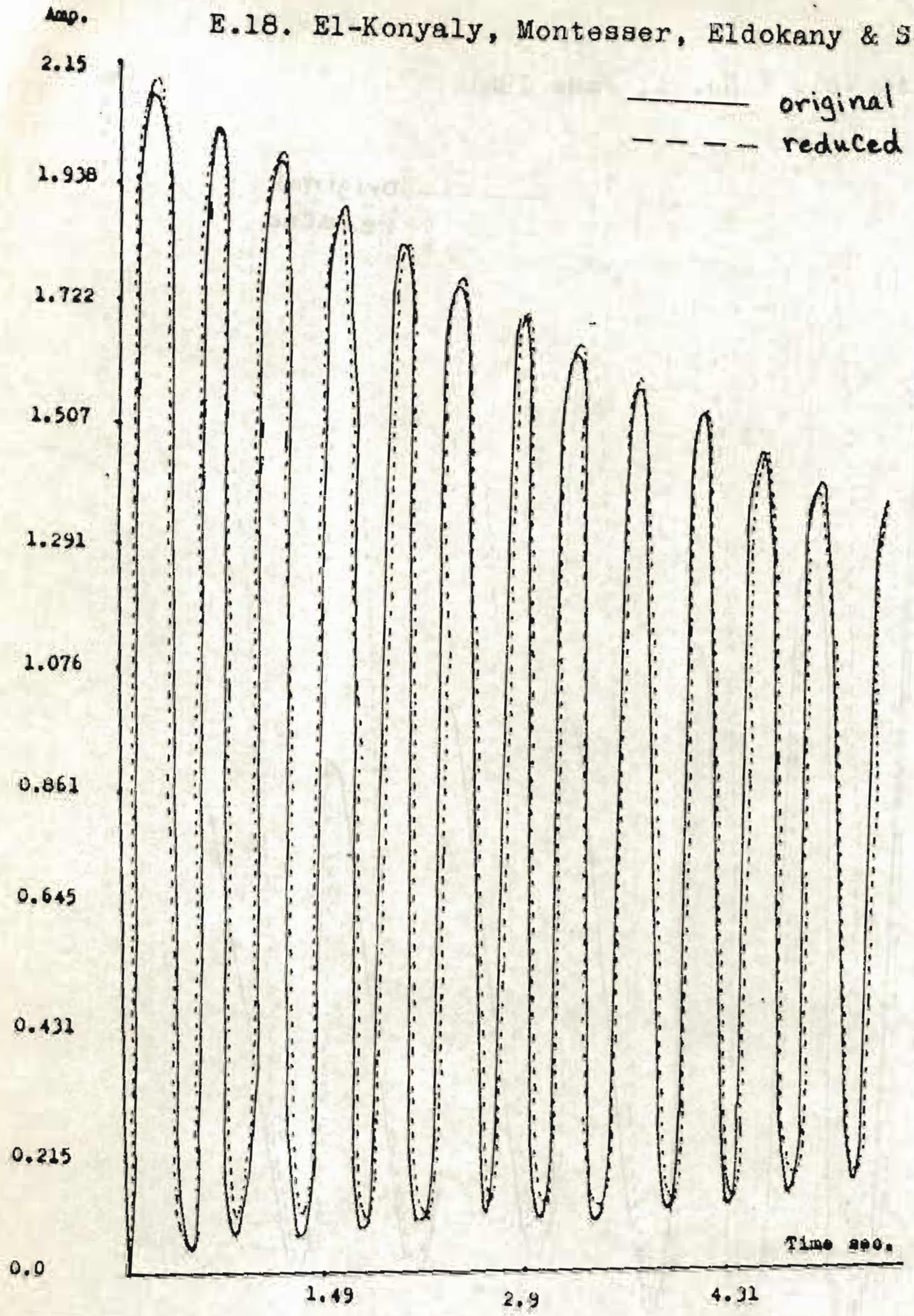

Fig. 2C Time response of state X_8 (Liapunov method).

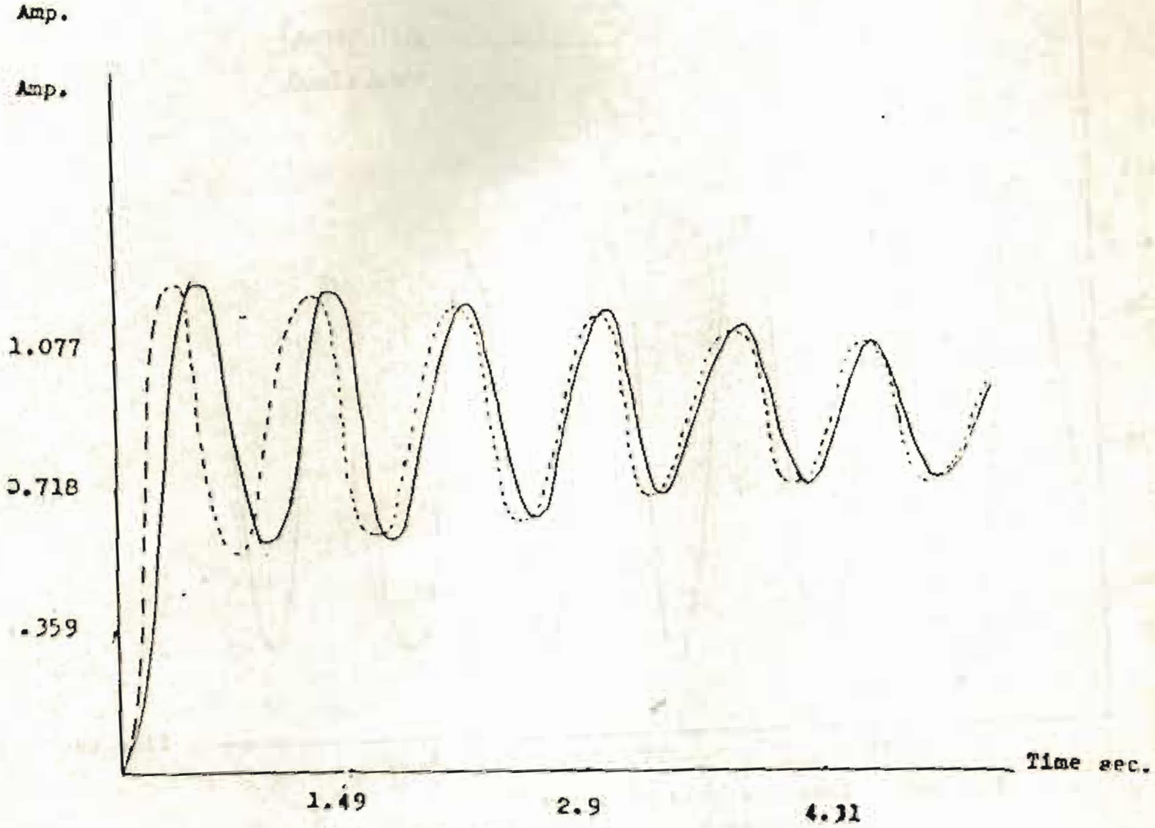

Fig. 2a Time response of state X_8 (Liapunov method).

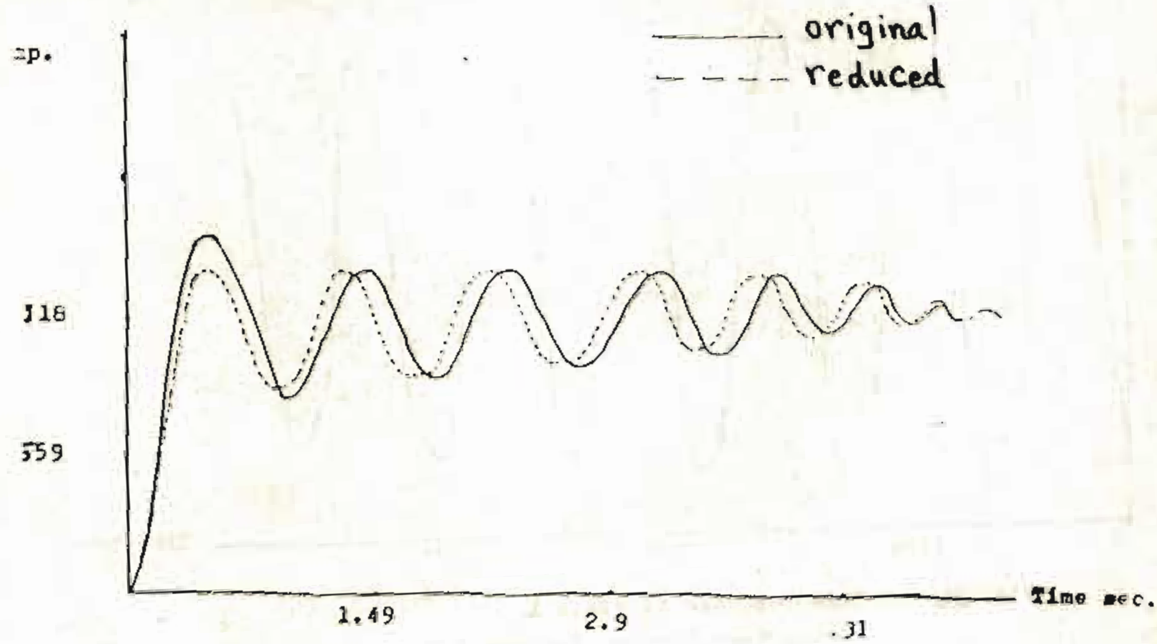

Fig. 2b Time response of state X_6 (Liapunov method).

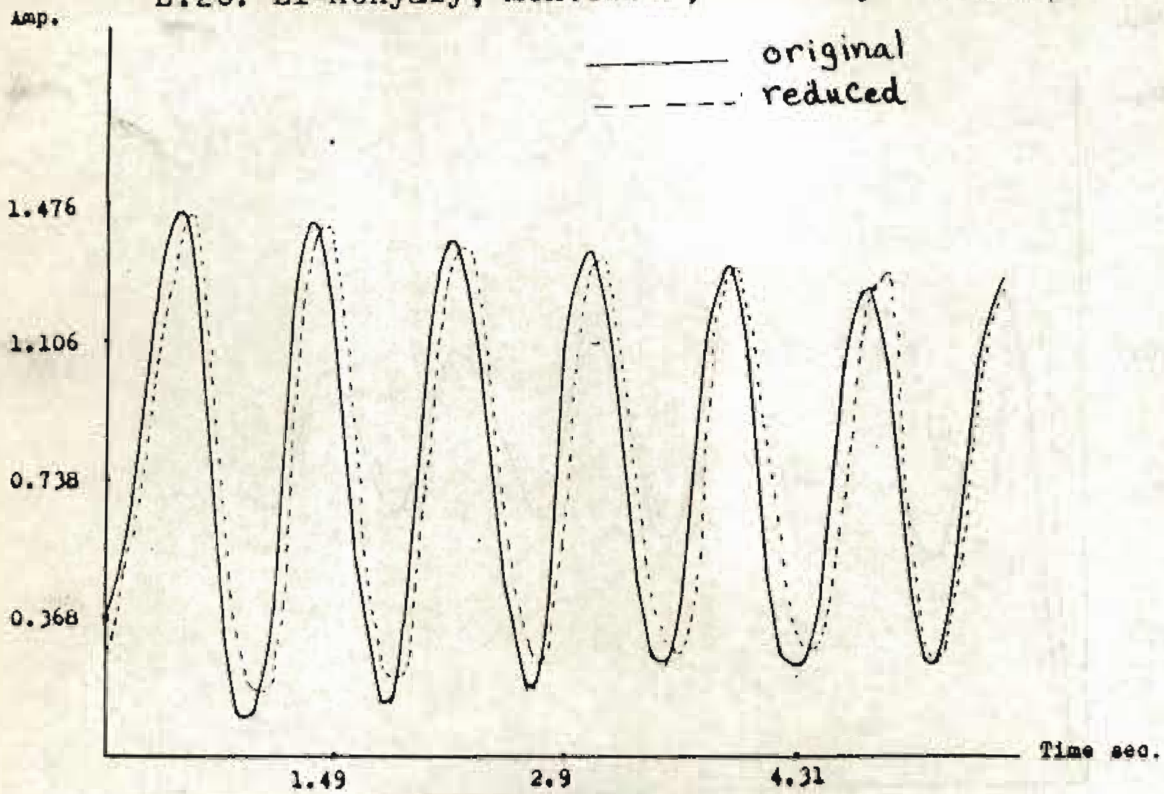

Fig. 3a Time response of state X_2
(Compalison between Davison and
Residualized methods).

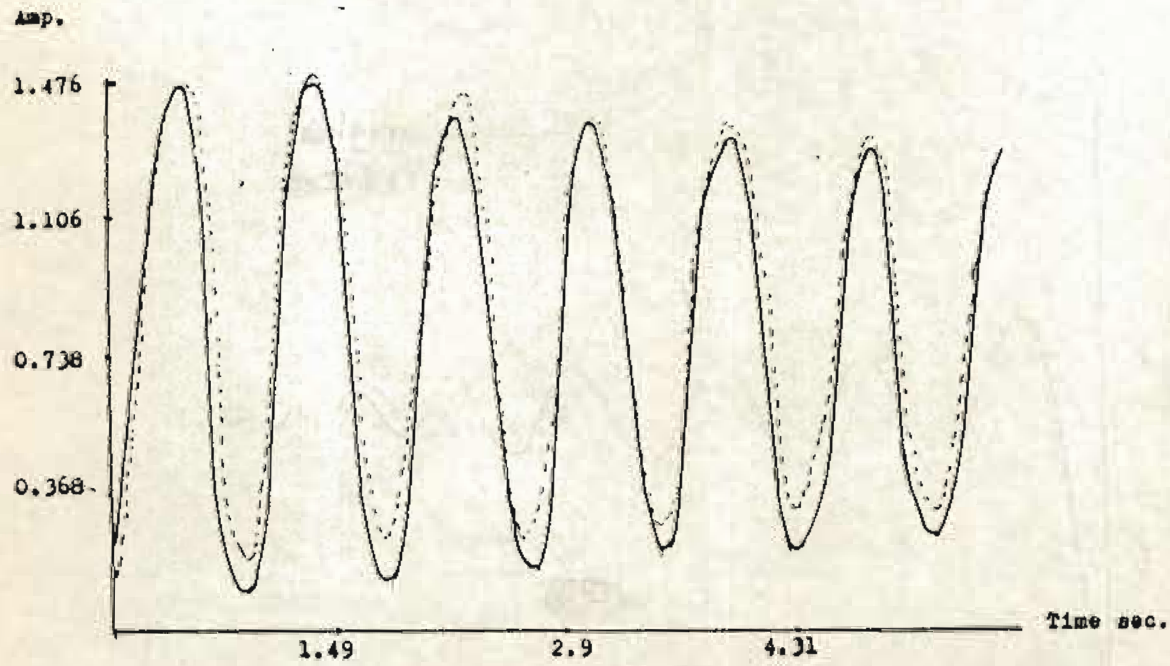

Fig. 3b Time response of state X_1
(Comparison between Davison and
Residualized methods).