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# The Importance of Interval Analytical Adjustment in Numerical-Geodetic Calculations.

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# THE IMPORTANCE OF INTERVAL ANALYTICAL ADJUSTMENT IN NUMERICAL-GEODETIC CALCULATIONS.

# اهمية الضبط التحليلي على فترات فى الحسابات الجيوديسية الرقمية

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الخلاصة :ـ

استحدث مع بداية القرن التاسع اسس تطبيق نظرية اقل مجموع المربعات لمعالجة الاخطاء العارضة طبقا لنظرية الاحتمالات من اجل الحصول على القيم الاكثر احتمالا للارصاد المساحية . وهذا البحث يعتبر اضافة جديدة كاحدى طرق الضبط التطبيقية المستخدمة فى ضبط الارصاد المساحية باستخدام الطريقة التحليلية باستخدام الفترات INTERVAL ANALYTICAL METHOD

وقد توصل البحث الى الحل الامثل باستخدام الحاسب الآلى من خلال تطبيق الطرق المختلفة للطريقة التحليلية باستخدام الفترات على مثال لارصاد ميزانية جيوديسية من اجل الحصول على القيم الاكثر احتمالا لهذه الارصاد .

ABSTRACT:

In this research different methods of the interval analytical adjustment have been introduced. Their quality has been tested by comparing the results obtained for a testing example " unidimensional geodetic level net ". It will be shown that an optimum solution can be obtained.

## INTRODUCTION:

The adjustment-calculations have been established at the beginning of the 19th century using the least squares method. It is assumed that, the accidental considered error follows the probability laws, and the most probable values and the mean error can be calculated by measuring other quantities.

If we renounce the probability theories and assume that the measurement error will not exceed a certain limit, so that for all considered values, a higher and a lower limit can be given. On using these intervals of the considered values in adjustment calculations, we will get also intervals of the unknown, which results from considering all the values between the given limits. The variety of the results based on the variety of the used data is an interval analytical question.

At the beginning of this research on the problem of interval analytical adjustment there is an introduction to the interval calculation. A computer language will be introduced which control the effect of round error. Linear interval equations will be introduced and the normal equations from the modified equation will be transformed to it. Different interval methods will be discussed for solving the normal interval equations and applied on a test example (geodetic level net).

It will be shown that an optimum solution can be obtained, based on the property of the M-Matrix of the normal equation matrix in geodetic nets.

To test the quality of results and the possibility of using the interval algorithms, the results will be compared with each other.

1-1 Basis of the interval calculation :-

R.E. Moore formulated in the 1960 s an Intervalanalysis which introduced a new numbers-element known as the Interval of real numbers [Nickel [1975] /11/, Schmitt [1977/16/]. His works constitute a main part in the interval mathematics. Hansen, kulisch, Nickel and others helped in the following years to develop this branch. The aim of introducing the interval-mathematics was to estimate the Round error in numerical calculations on computers using quantity-theoretical observations. Also to have under control its effect and accumulation on the algorithms.

Nickel [ 1966 ] /9/ asked for an errors-limit-arithmatics for computer, because an algorithm built only on approximate values without limits may be under certain cercomencies numerically and logically wrong.

There have been many trials in the last years to use interval-calculations in stastics and probability-calculations but till now there has been no success.

1-I. Definitions and Thesis:

This part presents the important definitions and thesis of the interval analysis [ BEECK/3/1971, ALEFELD /1/1971 which will be needed later on.

Defintion 1.1: A real closed interval [A]

$$[A] := [a_1 , a_2] = \{ a \in R \mid a_1 \leqslant a \leqslant a_2 \}$$

The quantity of all real closed intervals will be called I(R)

Defintion 1.2: Two Intervals [A] and [B] are equal

$$[A] = [B] \text{ when } a_1 = b_1 \wedge a_2 = b_2$$

Defintion 1.3: Connection of two intervals [A] $\ast$ [B] .

From I [R] by a calculation process $* \in \{ +, -, \cdot, : \}$

$$[A] * [B] := \{a * b \mid a_1 \leqslant a \leqslant a_2 , b_1 \leqslant b \leqslant b_2\}$$

From this definition:

ddition:

$$[a_1 , a_2] + [b_1 , b_2] = [a_1 + b_1 , a_2 + b_2]$$

Subtraction:

$$[a_1 , a_2] - [b_1 , b_2] = [a_1 - b_2 , a_2 - b_1]$$

Multiplication:

$$[a_1, a_2] \cdot [b_1, b_2] = [\min[a_1b_1, a_1b_2, a_2b_1, a_2b_2],$$
$$\max[a_1b_1, a_1b_2, a_2b_1, a_2b_2]]$$

Division:

$$[a_1, a_2] : [b_1, b_2] = [a_1, a_2] \cdot [1/b_2, 1/b_1]$$

Definition 1.4: If lower limit $a_1$ and upper limit $a_2$ are equal the interval [A] will be represented as point interval $\bar{A}$

$\bar{A}$: = [A] = [a , a] where $\bar{a} = a_1 = a_2$.

Corresponding to definition 1.4.

The addition will have the neutral element (0,0) on the multiplication will have the neutral element [1 , 1].

The special character of the structure I(R) will be shown by the following two thesis 1.1 and 1.2.

Thesis 1.1 subdistributivity in I (R)

Es gilt.

$[A] \cdot ([B] + [C]) \subseteq [A] \cdot [B] + [A] \cdot [C]$.

Thesis 1.2 partial quanties-property in [R].

when $[A] \subseteq [B]$ and $[C] \subseteq [D]$

it follows

$$[A] * [C] \subseteq [B] * [D]$$

Definition 1.5: Interval span ( [A] ) of the interval [A] from I [R] is the diference between the upper and lower limits:

$$SPAN([A]) := a_2 - a_1$$

The average formation with intervals will be taken as by quantities

Definition 1.6: The average of two intervals [A] and [B] from I [R] gives an interval containing real numbere which are elements from [A] and [B].

$$[A] \cap [B] := \{ C \mid C \in [A] \wedge C \in [B] \}$$

Assuming that $a_2 \geqslant b_1$ and $b_2 \geqslant a_1$, and the average

$[A] \cap [B]$ are not empty and gives [A] [B] =

$$[\max[a_1, b_1], \min[a_2, b_2]]$$

Definition 1.7: The amount of an interval [A] from I [A] corresponds to the amounts greatest limit.

$$|[A]| := \max(|a_1|, |a_2|)$$

The following part will show how the intervul analysis can be realised through a suitable interval arithmatic. This arithmatic has to get hold the round error automatically through machin arithmatic so that result interval contains surely the exact result.

## 1.2 The Realising of the intervalanalysis on the computer.

### 1.2.1 Machine interval arithmatic.

In a computer there is a limited quantity of numbers [machinenumbers] to be used for the quantity of all the real numbers R. To have the locking up property of the interval we have to approximate externally by the formulation and by every arithmatic operation [approximation of both the lower and the upper limit to the nearest machine-number]. This transformation from interval arithmatic into mashin artithmatic leads to numerical-result interval with greater span than those of the exact result intervals. The following increasing thesis [SCHMITT [1977] /16/ is valid :

Thesis 1.3 Increasing thesis

$$[A] * [B] \subseteq [[A] * [B]]_M \subseteq [A]_M * [B]_M$$

WHERE: $[A] * [B]$ = exact interval operation
$([A] * [B])_M$ = Machine operation affected by approximation between exact intervals.
$[A]_M * [B]_M$ = Machine operation between machine intervals.

There are many proposals for presenting intervals in computer.

| | | |
|---|---|---|
| $[\underline{x}, \bar{x}]$ | : | More, Kulisch<br>$\underline{X}$ = Lower limit, $\bar{X}$ = Upper limit. |
| $[\tilde{X}, \sigma]$ | : | Nickel<br>$\tilde{X}$ = mean value, $\sigma$ = symmetrical error limit<br>$\underline{X} = \tilde{X} - \sigma$ , $\bar{X} = \tilde{X} + \sigma$ |
| $[\tilde{X}; \underline{\sigma}, \bar{\sigma}]$ | : | Nickel for unsymmetrical limits<br>$\underline{X} = \tilde{X} - \underline{\sigma}$ , $\bar{X} = \tilde{X} + \bar{\sigma}$ |
| $[\underline{X}, \tilde{X}; \bar{X}]$ | : | Nickel [Triples-representation]<br>$\tilde{X}$ = mean value by the real arithmatic. |

The choosing of presenting method depends on the nature of the problem and the internal-storeplace-organization of the computer. Real interval-arithmatic can also be used in computers, but this will lead to the loss of the locking up property of the interval.
Therefore reals arithmatic can be used only for rough estimation of the round-error effect.

### 1.2.2. Computer-Interval Language

The program language has been developed on Elmansoura University to achieve control on the round error automatically by using the computer center [Nickel [1971]/10/, p. 19-22].

Definition 1.8 [by NICKEL]:

A: = [ a, ã , ā ] called triplex number where a ≤ ã ≤ a.ã is normally obtained by real calculations. ã is called the mean value or main value, it is "the most probable value" [ not the arithmatic mean].

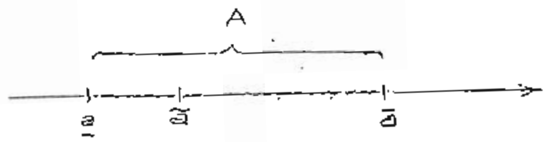


Fig. 1.1 Triplex number

A = [a , ã , ā]

Definition 1.9: Triplex constants will be written

A : = [ r , r̃ , r̄ ] where r , r̃ , r̄ real ;, r ≤ r̃ ≤ r̄.

Definition 1.10 Assuming A = [ a , ã , ā ].

| | | | |
|---|---|---|---|
| INF [A] | gives | a : = inf [A], | Resulttyp real |
| SUP [A] | gives | ā : = sup [A], | Resulttyp real |
| MAIN [A] | gives | ã : = mean value [A], | Resulttyp real |
| COMPOSE [r,s,t] | gives . | A : = [r,s,t], | Resulttyp triplex |

By COMPOSE the own parameters are real-values.Under conditions of r s t
By connecting variables in the Triplex-Program the priority rules have to be followed [ see Fig. 1.2 ].

| B | I | T | R |
|---|---|---|---|
| | T* | T | R |
| T | T | T | R |
| R | R | R | R |

where :
I = Integer
T = triplex
R = real

*) in case of the division T, otherwise I.

Fig. 1.2 Priority rules in Triplex-program.

## 2. Linear Interval Equations:

A linear real equation will have normally the form

$$A . X = b \qquad \ldots [2.1]$$

A is a real matrix, x and b real vectors.
Unidimensional and bidimeansional interval fields are defined in the following way:

Definition 2.1: If the coefficients of a matrix are real Intervals, it will be called Intervalmatrix. It will be written as follows:

$$[A] := [ [a_{jk}] ] \qquad i,k \in N$$
$$N = [ \text{ Quantity of natural numbers} ] .$$

Definition 2.2: If the coefficients of a vector are real interval, it will be called interval vector and will be written as follows:

$$[X] := [ [ X_k ] ] \qquad k \in N$$

Equations which have coefficients as real intervals will be called Inerval equations and have the form:

$$[A] . [X] = [b] \qquad \ldots [2.2]$$

### 2.1 Solutions of Interval equations:

BEECK [1971] /3/ found solutions for Interval equations assuming no singular interval matrix [A][det A≠0] for all $A \in [A]$ in 2.2 as solutions [Fig.2.1].

[ i ] The exact solution quantity [X] of the complex $[X] := [X \in R^n | A.\ x = b \text{ for } A \in [A], b \in [b]]$
The complex has a convex plyeder in n-dimensions space, and an unavailable arithmatic method and unsuitable to be transfered to a calculation computer center.
This leads to the locking up of the interval.

[ii] The optimum interval locking up solution, which is the Interval-cover
$[\hat{X}] := [ \inf [X] , \sup [X] ]$
inf[X] and sup [X] must not be elements of the quantity [X]

[iii] the Interval-upper cover
$[X] := [\inf [X] - |r_1|, \sup [X] + |r_2|] \supseteq [\hat{X}] \qquad r_1, r_2 \in R$

The numerical calculations gives always an upper cover because of the known rounding of the machine-interval arithmatic.

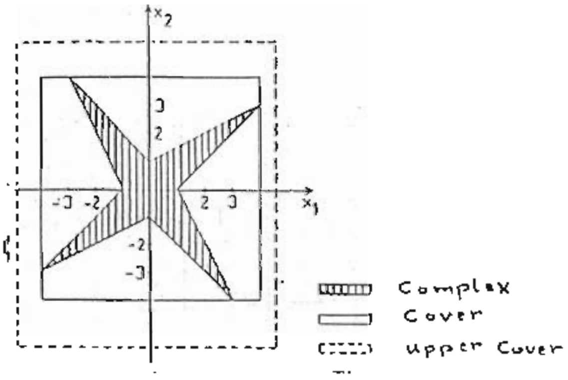



Fig. 2.1. Complex representation of the cover and the upper cover of

$$\left( \begin{bmatrix} 2,4 \\ -1,2 \end{bmatrix} \begin{bmatrix} -2,1 \\ 2,4 \end{bmatrix} \right) x = \left( \begin{bmatrix} -2,2 \\ -2,2 \end{bmatrix} \right)$$

### 2.2 Types of Error:

In numerical calculations there are three types of error [NUDING[1973]/13/:

1- Closed error: this is difference between [X] and $[\hat{X}]$.

2- Round error: which is not controlled in real arithmatic or through the known approximation of the machine-interval arithmatic which leads only to an upper cover as a solution.

3- Another closed error, because the algorithms lead to unrealistic great result intervals [upper cover] $[X] \supseteq [\hat{X}]$.

2.3 Experimental Investigations:

According to the task of this work there will be a difference between the outer solution

$$[ A . X = b \mid A \in [A], \quad b \in [b] ]$$

and the inner solution

$$[ X \mid A \in [A] :> A X \in [b] ]$$

The outer solution represents the quantities of all real equations A. X = b which fulfils the interval equation. The inner solution gives the quantities of the real vectors X which is fulfilled as interval vector [A] . [X] = [b] [ SCHMITT[1977] /16/:. Furthermore the two equation types can be written [A] . [X] = [b].

Once using an interval arithmatical Algorithm in a real equation the round error can be determined. The real coefficients of the starting system A. X = b will be transformed to an interval by approximation to the nearest machine number. In other case the [A] present interval matrix and [b] & [X] interval vectors, so the coefficients are real intervals.

Practical experience according to WONGWISES [1977] /17/ in treating linear equations with the Triplex arithmatic [chap. 1.2] proved that the Interval-Gauss-Elimination-method gave very pessimistic error span. The reason for this is that the I [R] is not a mathematical body.
Furthermore in greater equations i.e. absence of pivot-Element which does not contain the 0. In this case. the matrix will be known as "numerical singular" and the method collapses. This behaviour can be theoretically expected and quantitavely confirmed. In the last time there were many methods giving better limits. All are iteration methods. The most general form of the iteration form solution locking up of an interval equation [2.2] is called E = unity matrix

$$[ B^{-1} ] \approx [ A^{-1} ]$$

$$[ X^{[n+1]} ] = [ [E - [B^{-1}] [A] ] [ X^{[n]} ] + [B^{-1}] [b] ] \cap [X^{[n]} ] \quad \ldots \quad [2.3]$$

In all cases an approximate calculation of the inverse of A is required. Presupposition for the solution locking up is the presence of start interval, so that the solution contains $[X] \in [ X^{[0]} ]$. KRAWCZYK method is the most successful according to the numerical experience. Using this and other methods an interval analytical adiustment of unidimentional geodetic level nets will be carried out.

On this position it has to be mentioned that, principally all the states for unidimensional geodetic nets are valid for both level and gravimetric nets.

## 3. Test Example for level Net [Classic Adjustment]:

A level net has n=40 points and m=96 observed hight differences. The approximate hights of all the n points will be given, table 3.1.

All the m observations will be assumed having the weights 1.
The level net will be adjusted using the provided observations [REISSMANN [1976]/14/]. To get a no singular normal equation matrix, the present degree of freedem [vertical moving] has to be eliminated by fixing a point [ in this example point 1 ]. The obtained adjustment has a condition equation. To fulfil this condition we have to cancal the corresponding point [here point 1] from the residual equation. Accordingly there will be L = n-1 unknown from the residual equation of the form V = A.X -L [REISSMANN [1976], p.66 /14/] only the first six resedual equations will be written, [See Fig. 3.1, tables 3.1 and 3.2].

$$
\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}
=
\begin{bmatrix}
- dx_2 & + dx_3 & & \\
- dx_2 & & + dx_4 & \\
- dx_2 & & & \\
 & & - dx_4 & \\
 & & & + dx_5 \\
 & & - dx_4 & + dx_5 \\
 & & \cdot & \\
 & & \cdot & \\
 & & \cdot &
\end{bmatrix}
-
\begin{bmatrix} + 0{,}0130 \\ - 0{,}0050 \\ + 0{,}0100 \\ - 0{,}0150 \\ + 0{,}0030 \\ - 0{,}0140 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}
\quad [m]
$$

$$V \quad = \quad A \; . \; X \quad - \quad L$$

The adjustment of an test example (unidimensional geodetic level net) will be down using computer-program specially put for it according to a public Datei of the procedure solves a regular, symmetrical linear equation of the form A.y = r, where A a nxn-Matrix, y and r n-dimensional vectors. (The structural diagram of this computer program is shown in Appendix A). The results are given in Tab. 3.3

## 4. Interval analytical simplified adjustment of unidimensional nets :

The classic adjustment using the least squares method [REISSMANN [1976], p. 65/14/] starting from the modified resdual equation

$$V = A \, . \, X - L \qquad \ldots\ldots \qquad [4.1]$$

leads to a normal equation of the type

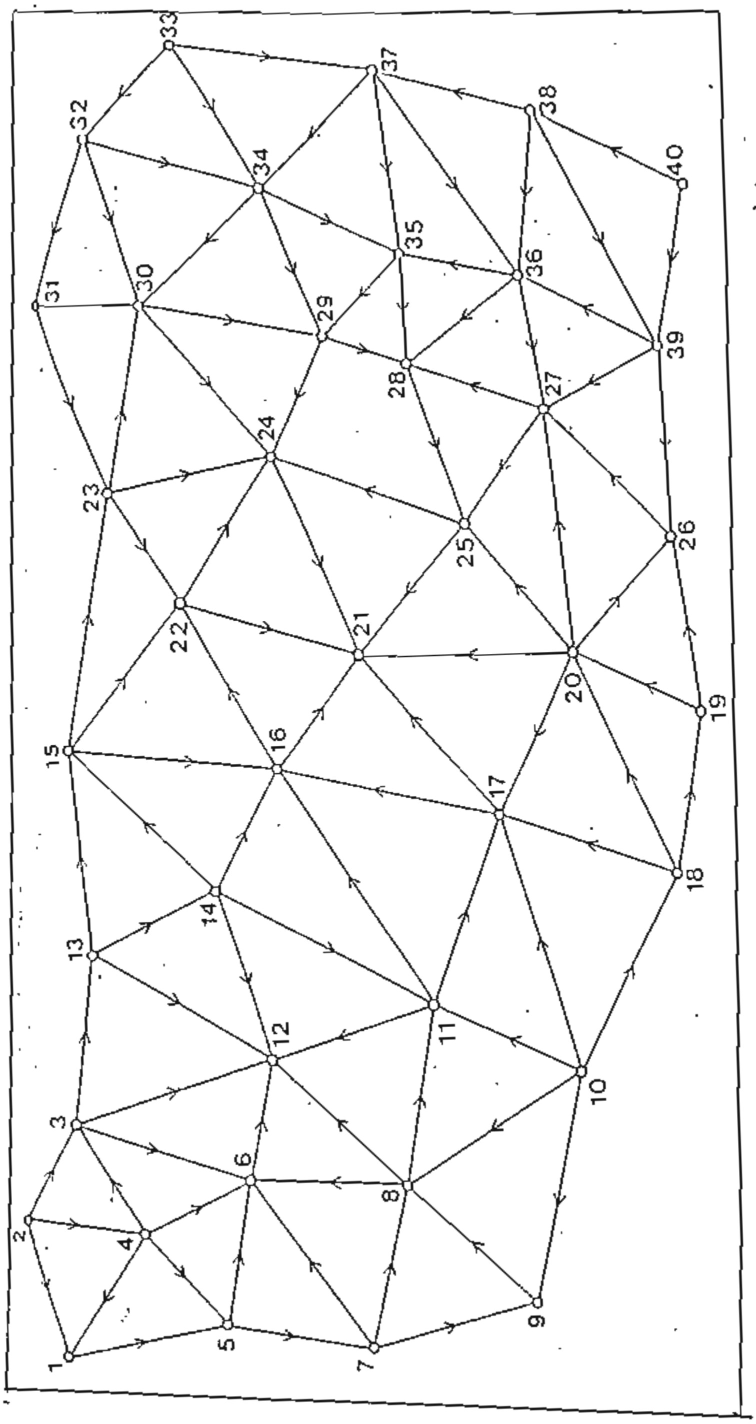
$$A^t \; P \; A \; X \; - \; A^t \; P \; L \; = \; 0 \qquad \ldots\ldots \qquad [4.2]$$

| Point number | Measured Level (m) |
|---|---|
| 1 | 9.3860 |
| 2 | 7.5470 |
| 3 | 10.0780 |
| 4 | 8.1280 |
| 5 | 15.9220 |
| 6 | 26.0450 |
| 7 | 17.9910 |
| 8 | 24.8400 |
| 9 | 18.4460 |
| 10 | 16.2430 |
| 11 | 26.9240 |
| 12 | 20.0650 |
| 13 | 15.1170 |
| 14 | 20.6550 |
| 15 | 23.6180 |
| 16 | 30.0400 |
| 17 | 28.4970 |
| 18 | 20.1660 |
| 19 | 21.9870 |
| 20 | 24.8420 |
| 21 | 40.1100 |
| 22 | 31.5610 |
| 23 | 25.3890 |
| 24 | 35.4650 |
| 25 | 31.6230 |
| 26 | 25.6320 |
| 27 | 27.0490 |
| 28 | 30.0910 |
| 29 | 29.7060 |
| 30 | 27.4810 |
| 31 | 24.2670 |
| 32 | 19.9050 |
| 33 | 15.5130 |
| 34 | 22.1280 |
| 35 | 25.7890 |
| 36 | 23.3340 |
| 37 | 20.4890 |
| 38 | 18.7970 |
| 39 | 19.4990 |
| 40 | 10.8450 |

Table 3.1

| Lines 1 – 48 | | | Lines 49 – 96 | | |
|---|---|---|---|---|---|
| From P. | To P. | Obs. (m) | From P. | To P. | Obs. (m) |
| 2 | 1 | 1.8500 | 16 | 21 | 10.0900 |
| 2 | 3 | 2.5240 | 22 | 21 | 0.5430 |
| 2 | 4 | 0.5090 | 24 | 21 | 4.6590 |
| 4 | 1 | 1.2430 | 25 | 21 | 8.4760 |
| 1 | 5 | 6.5390 | 16 | 22 | 1.5470 |
| 4 | 5 | 7.7000 | 15 | 22 | 7.9340 |
| 4 | 3 | 1.9580 | 23 | 22 | 6.1770 |
| 3 | 6 | 15.9610 | 22 | 24 | 3.9040 |
| 4 | 6 | 17.9050 | 15 | 23 | 1.7720 |
| 5 | 6 | 10.1050 | 31 | 23 | 1.1090 |
| 7 | 6 | 8.0740 | 23 | 30 | 2.1080 |
| 8 | 6 | 1.1920 | 23 | 24 | 10.0530 |
| 6 | 12 | 2.0290 | 30 | 24 | 7.9910 |
| 5 | 7 | 2.0500 | 29 | 24 | 5.7410 |
| 7 | 8 | 6.8580 | 25 | 24 | 3.0520 |
| 7 | 9 | 0.4440 | 20 | 25 | 1.5290 |
| 9 | 8 | 6.4140 | 27 | 25 | 4.5760 |
| 10 | 9 | 2.2030 | 27 | 28 | 3.0150 |
| 10 | 8 | 8.6030 | 36 | 27 | 3.7310 |
| 8 | 11 | 2.0720 | 39 | 27 | 7.5350 |
| 8 | 12 | 3.2320 | 26 | 27 | 1.4240 |
| 11 | 12 | 1.1250 | 19 | 26 | 3.6420 |
| 14 | 12 | 7.4340 | 39 | 26 | 6.1340 |
| 13 | 12 | 12.9100 | 39 | 36 | 3.8350 |
| 3 | 12 | 17.9950 | 38 | 39 | 0.7110 |
| 3 | 13 | 5.0230 | 40 | 39 | 8.6490 |
| 13 | 15 | 0.5100 | 36 | 28 | 6.7630 |
| 13 | 14 | 5.5250 | 36 | 35 | 2.4490 |
| 14 | 15 | 2.9730 | 37 | 36 | 2.8610 |
| 14 | 16 | 9.3560 | 38 | 36 | 4.5180 |
| 14 | 11 | 6.2020 | 35 | 28 | 4.3140 |
| 11 | 16 | 3.1100 | 29 | 28 | 6.3740 |
| 15 | 16 | 6.4420 | 30 | 29 | 2.2410 |
| 17 | 16 | 1.5260 | 34 | 29 | 7.5590 |
| 11 | 17 | 1.5060 | 35 | 29 | 3.9220 |
| 10 | 11 | 10.6660 | 34 | 35 | 3.6370 |
| 10 | 17 | 12.2840 | 37 | 34 | 1.6480 |
| 18 | 17 | 8.3190 | 33 | 34 | 6.6090 |
| 10 | 18 | 3.9230 | 32 | 34 | 2.2270 |
| 18 | 19 | 1.8060 | 34 | 30 | 5.3400 |
| 19 | 20 | 2.8610 | 31 | 30 | 3.2180 |
| 18 | 20 | 4.6680 | 32 | 30 | 7.5680 |
| 20 | 17 | 3.6680 | 32 | 31 | 4.3620 |
| 20 | 21 | 15.2420 | 33 | 32 | 4.3720 |
| 20 | 25 | 6.7910 | 33 | 37 | 4.9770 |
| 20 | 27 | 2.2040 | 37 | 35 | 5.2970 |
| 20 | 26 | 0.8040 | 38 | 37 | 1.7130 |
| 17 | 21 | 11.5910 | 40 | 38 | 7.9420 |

Table 3. 2.




o Nets - point

— Obs. difference in elevations

> Direction from lower to upper points

Fig. 3.1 : Test example: Unidimensional level net

where :

| | | |
|---|---|---|
| A | = | resedual equation matrix. |
| P | = | weight matrix |
| X | = | unknown vector |
| L | = | absolute vector |

The interval analytical adjustment starts in the following way :

4.1. The simplified adjustment with intervals:

It is well known that all measurements are accompanied by error arising from instrument, method, measured results, and the experience of the person. This error moves within certain limits [Defin. 1.1] which can be given for each observation as an interval containing the right values. The value of each interval will be determined and has to be fixed from variable to the other. Using this observation intervals [L] in adjustment calculations, the absolute member vector L in [4.1] will be transferred to [L], and we are looking for the inner solution of the cover[chap. 2.3]. According to definition 1.4 & 2.1, a matrix B with a point-interval coefficient, Point-interval-matrix, written $\dot{B}$. Accordingly, normal equation [4.1] transfer to the following linear interval equation:

$$A^t \; \dot{P} \; A \; [X] = A^t \; \dot{P} \; [L] \qquad \ldots\ldots\ldots \quad [4.3]$$

This equation is the most general representation of an interval normal equation and the starting point for determining the solution.Equation[4.3] is known as modified interval adjustment equation [SCHMITT [1977]/16/.

It is important to notice that : difference in weight of the different observations is unlogic and wrong, because only the reliability of the observations will be reflected in the value of the interval span,smaller interval span will give more exact measurements. P is identical to the unit matrix and [4.3] will be simplified to the following form :

$$A^t \cdot A \; [X] = \dot{A}^t \; [L] \qquad \ldots\ldots \quad [4.4]$$

To summarize it is noticed that: interval analytical adjustment is in fact an optimisation operation [BEECK [1971]/3/]. It renounces on considering the error according to the probability theories. By using observation-intervals, it gives intervals for the unknown, allowing the determination of the exactness of the unknown.

SCHMITT [1977]/16/ used successfully the interval analysis as alternative for direct solution of real equations by rounding-error estimations [unreal intervals [chap. 2.3]

The interval analysis has failed till now as alternative for error estimation of the break-through because of the too pessimistic results-intervals. It is hoped that this research [see chap. 4.4] will help in considering the interval analysis to be an essential part of the adjustment calculations by presenting the algorithm leading to the optimum solution locking up ; at least in the unidimensional geodetic nets.

## Classic adjustment

| Point- Nr. | measured H (m) | adjust. H (m) | mean square error (m) |
|---|---|---|---|
| 1 | 9.3860 | 9.3860 | 0.0000 |
| 2 | 7.5490 | 7.5454 | 0.0100 |
| 3 | 10.0780 | 10.0785 | 0.0111 |
| 4 | 8.1200 | 8.1346 | 0.0092 |
| 5 | 15.9220 | 15.9240 | 0.0099 |
| 6 | 26.0450 | 26.0363 | 0.0110 |
| 7 | 17.9910 | 17.9750 | 0.0124 |
| 8 | 24.8400 | 24.8381 | 0.0126 |
| 9 | 18.4460 | 18.4277 | 0.0143 |
| 10 | 16.2430 | 16.2371 | 0.0140 |
| 11 | 26.9240 | 26.9163 | 0.0135 |
| 12 | 28.0650 | 28.0573 | 0.0124 |
| 13 | 15.1170 | 15.1140 | 0.0135 |
| 14 | 20.6550 | 20.6396 | 0.0139 |
| 15 | 23.6100 | 23.6086 | 0.0146 |
| 16 | 30.0400 | 30.0200 | 0.0144 |
| 17 | 28.4970 | 28.5045 | 0.0145 |
| 18 | 20.1660 | 20.1759 | 0.0156 |
| 19 | 21.9870 | 21.9856 | 0.0171 |
| 20 | 24.8420 | 24.8422 | 0.0155 |
| 21 | 40.1100 | 40.0994 | 0.0152 |
| 22 | 31.5610 | 31.5545 | 0.0154 |
| 23 | 25.3890 | 25.3801 | 0.0160 |
| 24 | 35.4650 | 35.4534 | 0.0159 |
| 25 | 31.6230 | 31.6206 | 0.0162 |
| 26 | 25.6320 | 25.6358 | 0.0170 |
| 27 | 27.0490 | 27.0540 | 0.0165 |
| 28 | 30.0910 | 30.0862 | 0.0169 |
| 29 | 29.7060 | 29.7083 | 0.0169 |
| 30 | 27.4810 | 27.4755 | 0.0167 |
| 31 | 24.2670 | 24.2652 | 0.0178 |
| 32 | 19.9050 | 19.9051 | 0.0178 |
| 33 | 15.5130 | 15.5257 | 0.0185 |
| 34 | 22.1280 | 22.1390 | 0.0172 |
| 35 | 25.7890 | 25.7804 | 0.0173 |
| 36 | 23.3340 | 23.3306 | 0.0171 |
| 37 | 20.4890 | 20.4910 | 0.0176 |
| 38 | 18.7970 | 18.7955 | 0.0182 |
| 39 | 19.4990 | 19.5053 | 0.0175 |
| 40 | 10.8450 | 10.8549 | 0.0198 |

$m_0 = 0.0137$ m

Fig. 3.3

4.1.1, choosing the observations-intervals in the test example:

In classic adjustment, the mean square error of the observation $L_i$ will be calculated

$$m_i = m_0 \; / \; \sqrt{P_i} \qquad \ldots\ldots \quad [4.6]$$

$m_0$ = the mean square error of the unit weight

$P_i$ = weight of the observation $L_i$.

In the test example the true value of a certain observation $L_i$ lies between $[L_i - m_i]$ to $[L_i + m_i]$

The interval span is $[2 \cdot m_i]$, assuming $P_i = 1$ for all observations $L_i$. For the test example [Triplex-representation] chap. 1.2.1, we recieve the observation-Interval

$$[\, L_i - m_0 \, , \, L_i \, , \, L_i + m_0 \,] \, . \qquad \ldots\ldots \quad [4.6]$$

From 4.6 we obtain the absolute member interval for a reseidual equation $V_i$ [Triplex-representation] ,

$$[\, l_i - m_0 \, , \, l_i \, , \, l_i + m_0 \,] \qquad \ldots\ldots \quad [4.7]$$

The absolute member interval is symmetrical according to the observations intervals. For example we present the Ist 6 members of the absolute member interval vectors of the test example in example 4.1, compare example 3.1.

Example 4.1

$$\begin{bmatrix} -\,0{,}0007 & , & +\,0{,}0130 & , & +\,0{,}0267 \\ -\,0{,}0187 & , & -\,0{,}0050 & , & +\,0{,}0087 \\ -\,0{,}0037 & , & +\,0{,}0100 & , & +\,0{,}0237 \\ -\,0{,}0287 & , & -\,0{,}0150 & , & -\,0{,}0013 \\ -\,0{,}0107 & , & +\,0{,}0030 & , & +\,0{,}0167 \\ -\,0{,}0277 & , & -\,0{,}0140 & , & -\,0{,}0003 \\ \vdots & & \vdots & & \vdots \end{bmatrix} = [L]$$

[m]

4.1.2. Criterion for comparison between classic and interval adjustment:

The accuracy of the used interval method will be tested by comparing the result interval of the interval adjustment with the corresponding mean . . error interval from classic adjustment, for each unknown. The mean error interval will be fixed as follows (Triplex representation):

$$[\, X_i - m_{X_i},\ X_i,\ X_i + m_{X_i} \,] \qquad \ldots\ldots [4.8]$$

where :

$X_i$ = adjusted unknown

$m_{X_i}$ = mean square error of the adjusted unknown.

The mean error interval will be contain the best fit value of the unknown only according to a certain probability. On the other hand, the results interval will surely contain the best fit value of the unknown assuming that the observations are within the observations-intervals.
Best results of comparison can be obtained on comparing the span of results interval and the mean error interval.

## 4.2 Interval adjustment using the left inverted matrix, also comparing the results with the classic adjustment results:

The method has been proposed by BRUNKE [1977]/5/ and tested on a EDM-net. To avoid the dependant intervals [Chap.2.2] and the increase of the result-span Brunke used a real matrix which will be multiplied with the absolute member-interval-vector [l] to give the result.
Solving an interval-normal equation of the form [4.4] by using the matrix rules to find [X]

$$[\, X \,] = [\, [A^t A]^{-1} A^t ] \, [l] \qquad \ldots\ldots [4.9]$$

BRUNKE [1977], p.30/5/ used the following definition:

*Definition 4.1:* A is a real mxn-Matrix and $n \leq m$ and R [A] = R $[A^t A]$ = n, R [A] = Range of the matrix A]. The matrix $A_L^{-1} := [A^t A]^{-1} A^t$ which is the left inverted matrix of A and $A_L^{-1} A = E$ [E = unity matrix] from 4.9 $[A^t A]^{-1} A^t$ is the required matrix, and the computer-Programs have been put to carry out the solution.

### 4.2.1 Solution Using Triplex-Intervals.

To get the lock-up property of the solution [X] Triplex arithmatic has been used [comp.1.2]. [L] consits of $\underline{L}$ lower limit, $\ddot{L}$ mean value and $\bar{L}$ the higher limit, it will be formed according to 4.7, (is used the Triplex-program)

The calculation as follows :

[ i ] - Fixing point 1 leads to reduction of the unknowns

l to l = n - 1

The Triplex - vector [l] will be calculated from l and the mean square error of the weight unity

[ii] - $A^tP$ and $A^{-1}_L = [A^t PA]^{-1} A^t P$
will be determined at P = E

[iii] - Using the multiplication $A^{-1}_L$ with [l] to get the Triplexsolution [x].

[iv] - Using the machine - interval - arithmatic [1 - 2 - 1], we get an approximation with which [x] presents an interval upper cover containing the solution.

On adding the approximate levels to [X] we get the final value of the unknown $x_i$ as triplex number $x_i$. A triplex unknown x gives $\underline{x}$ the lower limit $\tilde{x}$ the most probable value and $\bar{x}$ the upper limit of the unknown x.
The results are found in table 4.1.
From table 4.1 it is clear that the mean value of the unknown is equal to the most probable value of the unknown from the classic adjustment (see def. 1.8, compare Tab.3.3 and Compare 4.11).

To compare the quality of the method used, the spans of the unknown from the interval adjustment has to be compared with mean error interval as in Tab. 4.2. It is clear that the span of the unknown has a mean value of seven times the mean error-interval and ranges between 3,32 (at point 4) and 8.87 (at point 39). This way does not prevent the dependence of the intervals in the calculation. In order to see, the value of increase in the span due to external approximation, the same solution

## INTERVAL ADJUSTMGNT

| Point Nr. | Lower limit (m) | adjusted value (m) | upper limit (m) | span dx (m) |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 |
| 1 | 9.3859 | 9.3860 | 9.3860 | 0.0000 |
| 2 | 7.5099 | 7.5454 | 7.5810 | 0.0708 |
| 3 | 10.0277 | 10.0785 | 10.1295 | 0.1016 |
| 4 | 8.1041 | 8.1346 | 8.1652 | 0.0609 |
| 5 | 15.8849 | 15.9240 | 15.9630 | 0.0779 |
| 6 | 25.9075 | 26.0363 | 26.0851 | 0.0974 |
| 7 | 17.9141 | 17.9750 | 18.0362 | 0.1219 |
| 8 | 24.7651 | 24.8381 | 24.9109 | 0.1455 |
| 9 | 18.3442 | 18.4277 | 18.5111 | 0.1665 |
| 10 | 16.1460 | 16.2371 | 16.3287 | 0.1824 |
| 11 | 26.8317 | 26.9163 | 27.0010 | 0.1689 |
| 12 | 27.9969 | 28.0593 | 28.1219 | 0.1248 |
| 13 | 15.0341 | 15.1140 | 15.1942 | 0.1597 |
| 14 | 20.5529 | 20.6396 | 20.7266 | 0.1733 |
| 15 | 23.5102 | 23.6086 | 23.7073 | 0.1968 |
| 16 | 29.9230 | 30.0201 | 30.1165 | 0.1931 |
| 17 | 28.4048 | 28.5045 | 28.6039 | 0.1987 |
| 18 | 20.0642 | 20.1759 | 20.2870 | 0.2224 |
| 19 | 21.8567 | 21.9856 | 22.1151 | 0.2578 |
| 20 | 24.7238 | 24.8422 | 24.9610 | 0.2368 |
| 21 | 39.9902 | 40.0994 | 40.2086 | 0.2180 |
| 22 | 31.4453 | 31.5545 | 31.6634 | 0.2178 |
| 23 | 25.2575 | 25.3801 | 25.5029 | 0.2451 |
| 24 | 35.3323 | 35.4534 | 35.5743 | 0.2417 |
| 25 | 31.4763 | 31.6206 | 31.7454 | 0.2480 |
| 26 | 25.4941 | 25.6356 | 25.7774 | 0.2827 |
| 27 | 26.9148 | 27.0540 | 27.1937 | 0.2783 |
| 28 | 29.9436 | 30.0862 | 30.2283 | 0.2842 |
| 29 | 29.5667 | 29.7083 | 29.8505 | 0.2832 |
| 30 | 27.3373 | 27.4755 | 27.6138 | 0.2759 |
| 31 | 24.1221 | 24.2652 | 24.4084 | 0.2856 |
| 32 | 19.7547 | 19.9051 | 20.0551 | 0.2998 |
| 33 | 15.3704 | 15.5257 | 15.6816 | 0.3105 |
| 34 | 21.9901 | 22.1390 | 22.2885 | 0.2979 |
| 35 | 25.6300 | 25.7804 | 25.9304 | 0.2998 |
| 36 | 23.1006 | 23.3306 | 23.4005 | 0.2993 |
| 37 | 20.3362 | 20.4918 | 20.6464 | 0.3096 |
| 38 | 18.6362 | 18.7955 | 18.9542 | 0.3174 |
| 39 | 19.3525 | 19.5053 | 19.6577 | 0.3047 |
| 40 | 10.6903 | 10.8549 | 11.0190 | 0.3281 |

span of observations = 0.0274 (m)

Table 4.1

| | Interval – Adjustment | Classic Adjustment | |
|---|---|---|---|
| Point Nr. | Span (x) (m) | Span ($m_x$) (m) | Span (x)/Span ($m_x$) |
| 1 | 2 | 3 | 4 |
| 1 | 0,0001 | 0,0000 | – |
| 2 | 0,0711 | 0,0200 | 3,56 |
| 3 | 0,1018 | 0,0222 | 4,87 |
| 4 | 0,0611 | 0,0184 | 3,32 |
| 5 | 0,0761 | 0,0198 | 3,94 |
| 6 | 0,0976 | 0,0220 | 4,44 |
| 7 | 0,1221 | 0,0248 | 4,92 |
| 8 | 0,1458 | 0,0252 | 5,79 |
| 9 | 0,1669 | 0,0286 | 5,84 |
| 10 | 0,1827 | 0,0280 | 6,53 |
| 11 | 0,1693 | 0,0270 | 6,27 |
| 12 | 0,1250 | 0,0248 | 5,04 |
| 13 | 0,1601 | 0,0270 | 5,93 |
| 14 | 0,1737 | 0,0278 | 6,25 |
| 15 | 0,1971 | 0,0292 | 6,75 |
| 16 | 0,1935 | 0,0288 | 6,72 |
| 17 | 0,1991 | 0,0290 | 6,87 |
| 18 | 0,2228 | 0,0312 | 7,14 |
| 19 | 0,2584 | 0,0342 | 7,56 |
| 20 | 0,2372 | 0,0310 | 7,65 |
| 21 | 0,2184 | 0,0304 | 7,18 |
| 22 | 0,2181 | 0,0308 | 7,08 |
| 23 | 0,2454 | 0,0320 | 7,67 |
| 24 | 0,2420 | 0,0318 | 7,61 |
| 25 | 0,2491 | 0,0324 | 7,69 |
| 26 | 0,2833 | 0,0340 | 8,33 |
| 27 | 0,2789 | 0,0330 | 8,45 |
| 28 | 0,2847 | 0,0338 | 8,42 |
| 29 | 0,2838 | 0,0338 | 8,40 |
| 30 | 0,2765 | 0,0334 | 8,28 |
| 31 | 0,2863 | 0,0356 | 7,39 |
| 32 | 0,3004 | 0,0356 | 8,44 |
| 33 | 0,3112 | 0,0370 | 8,41 |
| 34 | 0,2984 | 0,0344 | 8,61 |
| 35 | 0,3004 | 0,0346 | 8,68 |
| 36 | 0,2999 | 0,0342 | 8,77 |
| 37 | 0,3102 | 0,0352 | 8,81 |
| 38 | 0,3180 | 0,0364 | 8,74 |
| 39 | 0,3052 | 0,0344 | 8,87 |
| 40 | 0,3287 | 0,0396 | 8,30 |

Table 4.2

Mean : 7,01
—— : Minimum value
==== : Maximum value

has been repeated using real interval arithmatic.

#### 4.2.2. Solution using Real Intervals:

The absolute-member-vector [L] will be replaced through three real vectors:

$L_u$ containing the lower limit

L the most probale value

$L_u$ upper limit $L_2$ of the absolute member-vector.

These calculations do not allow any approximation. Table 4.3 presents the results of the test example.

Comparing these results with those in table 4.2 we find that:

- On using real interval arithmatic the span is smaller to the factor 0.92 [point 40] and 0,72 [point 2] compared with the Triplex-calculations.
- The external approximations increased the span in comparison with real interval-arithmatic results with about 18%.

To summerize we find that:

The interval calculation through the left inverted-matrix does not eradicate the effect of the dependence of the intervals. A great interval-uppercover for the adjusted unknown which has a mean value of seven times that of the mean error-interval of the classic adjustment has been obtained.
The span increases as the distance from the fixed point increases, and this is another indication of the explained dependence. From these fact it is clear that the interval adjustment through these left-inverted-matrix is not suitable to be applied on unidimensional geodetic nets. For better results different methods have been experimentally examined [NICKEL [1978]/12/.

#### 4.3 Interval adjustment by the krawczyk-method and comparison with the classic adjustment of the test exampe:

The krawczyk method is suitable for solving linear interval equations. Therefore it can lead to the solution locking up of the interval-normal-equation [see eq.4.3].

#### 4.3.1. krawczyk method:

This is an Iterations method presented by WONGWISES [1977]/17/ for solving systems of the form

$$A[X] = b \qquad \ldots\ldots [4.10]$$

The iterations steps are:

$$[X^{[n+1]}] = [[E-B^{-1}A][X^{[n]}] + B^{-1}b] \cap [X^{[n]}] \qquad \ldots\ldots [4.11]$$

where $E$ = Unity matrix E

$B^{-1} \approx A^{-1}$ [ compare [2.3] ]

## INTERVAL ADJUSTMENT

(with real arithmatics)

| Point Nr. | Lower limit (m) | Adjusted Value (m) | Upper limit (m) | Span (m) |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 |
| 1 | 9.3060 | 9.3060 | 9.3060 | 0.0000 |
| 2 | 7.5164 | 7.5154 | 7.5744 | 0.0580 |
| 3 | 10.0360 | 10.0785 | 10.1211 | 0.0852 |
| 4 | 0.1090 | 0.1346 | 0.1602 | 0.0512 |
| 5 | 15.8927 | 15.9240 | 15.9554 | 0.0627 |
| 6 | 25.9961 | 26.0363 | 26.0765 | 0.0804 |
| 7 | 17.9212 | 17.9750 | 18.0288 | 0.1076 |
| 8 | 24.7779 | 24.8389 | 24.8998 | 0.1219 |
| 9 | 18.3570 | 18.4277 | 18.4985 | 0.1415 |
| 10 | 16.1573 | 16.2371 | 16.3169 | 0.1596 |
| 11 | 26.8467 | 26.9163 | 26.9859 | 0.1392 |
| 12 | 23.0048 | 23.0593 | 23.1138 | 0.1090 |
| 13 | 15.1962 | 15.1190 | 15.1318 | 0.1356 |
| 14 | 20.5658 | 20.6396 | 20.7135 | 0.1477 |
| 15 | 23.5200 | 23.6086 | 23.6973 | 0.1773 |
| 16 | 29.9362 | 30.0200 | 30.1037 | 0.1675 |
| 17 | 28.4163 | 28.5045 | 28.5927 | 0.1764 |
| 18 | 20.0735 | 20.1754 | 20.2779 | 0.2091 |
| 19 | 21.8679 | 21.9856 | 22.1038 | 0.2364 |
| 20 | 24.7332 | 24.8422 | 24.9508 | 0.2170 |
| 21 | 40.0021 | 40.0994 | 40.1967 | 0.1946 |
| 22 | 31.4551 | 31.5595 | 31.6539 | 0.1988 |
| 23 | 25.2676 | 25.3801 | 25.4925 | 0.2250 |
| 24 | 35.3425 | 35.4534 | 35.5643 | 0.2219 |
| 25 | 31.5060 | 31.6206 | 31.7353 | 0.2292 |
| 26 | 25.5184 | 25.6350 | 25.7632 | 0.2448 |
| 27 | 26.9277 | 27.0540 | 27.1802 | 0.2525 |
| 28 | 29.9559 | 30.0862 | 30.2165 | 0.2606 |
| 29 | 29.5792 | 29.7083 | 29.8375 | 0.2583 |
| 30 | 27.3496 | 27.4755 | 27.6015 | 0.2519 |
| 31 | 24.1332 | 24.2652 | 24.3973 | 0.2641 |
| 32 | 19.7666 | 19.9051 | 20.0437 | 0.2771 |
| 33 | 15.3822 | 15.5257 | 15.6693 | 0.2871 |
| 34 | 22.0022 | 22.1390 | 22.2759 | 0.2737 |
| 35 | 25.6433 | 25.7804 | 25.9175 | 0.2742 |
| 36 | 23.1928 | 23.3306 | 23.4683 | 0.2755 |
| 37 | 20.3490 | 20.4710 | 20.6331 | 0.2841 |
| 38 | 18.6509 | 18.7955 | 18.9409 | 0.2900 |
| 39 | 19.3667 | 19.5053 | 19.6440 | 0.2773 |
| 40 | 10.7040 | 10.8549 | 11.0058 | 0.3018 |

Span of observations = 0.0274 (m)

Table 4. 3

Thus, there are three important questions:

[ i ] How to obtain the value of $[X^{[0]}]$ ?

[ii] Under which conditions is the method convergent, so that results locking up $[X^{[0]}]$ has severe monoton interval vector series:

$$[x^{[0]}] \supset [X^{[1]}] \supset [X^{[2]}] \quad .....$$

which converges at the interval cover ?

[iii] How can $B^{-1} = B^{-1}$ ?

The solution in krawczyk method will be as follows :

- Opposit to other methods, $B^{-1}$ will be used for $A^{-1}$ and not $[B^{-1}]$
- The method converges when the matrix norm (matrix-norm of matrix A:

$$\| A \| : = \max_{i = 1(1)n} \sum_{k=1}^{n} | a_{ik} |$$

$$\| E - B^{-1} A \| < 1 \qquad ..... (4.12)$$

To avoid increase in interval which leads to the numerical singularity a "defect güessing" is used. The defect materix R defined as

$R : = E - B^{-1} A$ we get

$$[X^{[0]}] = [-1,1] \cdot \frac{\| B^{-1} [b - A \tilde{X}] \|}{1 - \| R \|} + \tilde{X} \qquad ..... (4.13)$$

$$\text{where } \tilde{X} \in [X] : = B^{-1} b \qquad ..... (4.14)$$

If more details explanations are required refer to WONGWISES [1977] / 17 / . A triplex program has been formulated for this method which consists of four procedures.

4.3.2. Interval adjustment by the krawczyk-method :

To apply the krawczyk method on an interval-equation of the form

$$A^{t} \cdot p^{\wedge} [X] = A^{t} p [L]$$

which has absolute member-vector assumed to be real interval vector, we have to replace the b with a real interval vector [b] by changing the procedure. Otherwise, the procedure of WONGWISES/17/ [1977] has been slightly changed. The required data has been taken from the test example. Table 4.4 present the results obtained.

- Comparing these results with those in table 3.3 shows that the mean value of the results interval is not identical with the most probable value of the classic adjustment. The maximum difference between mean value and the most probable value of the classic adjustment is 0.0102 m (point 9). These difference are of accidental nature. Therefore it is better not to use the mean value as the most probale value. It is easy to examine if the result interval contains the most probable value of the classic adjustment.

These result intervals are symmetrical corresponding to the symmetrical exit intervals.

On comparing the span dx [column 5] with the result Interval-span it is formed as follows :

Upper limit [column 4]-Lower limit [column 2], we find that the result span is maximum 0.003 m [point 6 & 11] greater than the span dx [Tab. 4.1 & 4.2]

The mean dx is smaller than the dx in the left inverted solution which may be attributed to the effect of the approximation in each calculation which may be dependent on the size of the interval span.

The result span is compared in table 4.5 with the mean-error interval.
By examination of the span in tables 4.5 and 4.4 we find that:

The span [X] does not increase with increasing the distance from the fixed point. On beginning from the test example with an observation-interval-span = 0,0274 for all observations, we get for point 2 the greatest interval span 0,0807 m and for point 40 the smallest interval span 0279 m. Correspondingly, the spans [X] are maximum 4 times [point 2] and minimum 0,7 times [point 40] greater than the span [mx]. The spans [X] increase with respect to the mean-error-interval by the factor 2. We have to pay attention that, the result interval contains the real values, while in the classic adjustment will be within probability [< 1] for the confidence interval.

The unrequired dependence of the intervals followed by increase in the result span on using the krawczyk method is smaller in applying it to the test example compared with that obtained by using the left-inverted solution.

4.4 Optimum interval locking up of the interval normal equation using the M-Matrix property, and comparison with the classic adjustment in case of the test example:

Every normal equation matrix $A^t\ p\ A$ of unidimensional net shows the following sign distribution in the modified adjustment:

$$\begin{vmatrix} + & - & - & \cdot & - \\ - & + & - & \cdot & - \\ - & - & + & \cdot & - \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ - & - & - & - & + \end{vmatrix}$$

Example 4.1: sign distribution of the normal equation matrix. Applying on the test example will have:

$$\begin{vmatrix} 3 & -1 & -1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 5 & -1 & 0 & -1 & 0 & \dots & 0 \\ 0 & 0 & 5 & -1 & -1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 4 & -1 & -1 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 2 \end{vmatrix}$$

Example 4.2 : Normal equation matrix of the test example. Moreover the normal equation matrix presents according to Varga a defined M-Matrix which is defined as follows [BARTH [1974]/. 2/].

# INTERVAL ADJUSTMENT

## (KRAWCZYK)

| Point Nr. | Lower limit (m) | Mean (m) | Upper limit (m) | Span (dx)(m) |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 |
| 1 | 9.3659 | 9.3860 | 9.3860 | 0.0000 |
| 2 | 7.5078 | 7.5401 | 7.5885 | 0.0806 |
| 3 | 10.0422 | 10.0813 | 10.1203 | 0.0777 |
| 4 | 8.1036 | 8.1334 | 8.1633 | 0.0596 |
| 5 | 15.8899 | 15.9223 | 15.9546 | 0.0645 |
| 6 | 26.0080 | 26.0403 | 26.0725 | 0.0642 |
| 7 | 17.9507 | 17.9812 | 18.0116 | 0.0605 |
| 8 | 24.0087 | 24.0459 | 24.0829 | 0.0740 |
| 9 | 18.4101 | 18.4379 | 18.4656 | 0.0553 |
| 10 | 16.2101 | 16.2453 | 16.2804 | 0.0701 |
| 11 | 26.8889 | 26.9237 | 26.9583 | 0.0691 |
| 12 | 28.0390 | 28.0643 | 28.0809 | 0.0490 |
| 13 | 15.0873 | 15.1181 | 15.1490 | 0.0616 |
| 14 | 20.6190 | 20.6480 | 20.6762 | 0.0563 |
| 15 | 23.5835 | 23.6167 | 23.6498 | 0.0662 |
| 16 | 30.0006 | 30.0291 | 30.0576 | 0.0569 |
| 17 | 28.4817 | 28.5109 | 28.5401 | 0.0582 |
| 18 | 20.1445 | 20.1746 | 20.2048 | 0.0602 |
| 19 | 21.9492 | 21.9826 | 22.0161 | 0.0667 |
| 20 | 24.8117 | 24.8421 | 24.8725 | 0.0607 |
| 21 | 40.0772 | 40.1056 | 40.1343 | 0.0569 |
| 22 | 31.5373 | 31.5642 | 31.5912 | 0.0537 |
| 23 | 25.3569 | 25.3872 | 25.4176 | 0.0605 |
| 24 | 35.4317 | 35.4601 | 35.4883 | 0.0563 |
| 25 | 31.5980 | 31.6241 | 31.6503 | 0.0521 |
| 26 | 25.6098 | 25.6352 | 25.6606 | 0.0507 |
| 27 | 27.0265 | 27.0531 | 27.0796 | 0.0529 |
| 28 | 30.0570 | 30.0864 | 30.1159 | 0.0587 |
| 29 | 29.6830 | 29.7112 | 29.7394 | 0.0563 |
| 30 | 27.4534 | 27.4793 | 27.5053 | 0.0518 |
| 31 | 24.2474 | 24.2702 | 24.2930 | 0.0454 |
| 32 | 19.8800 | 19.9059 | 19.9319 | 0.0518 |
| 33 | 15.4971 | 15.5240 | 15.5511 | 0.0538 |
| 34 | 22.1159 | 22.1367 | 22.1579 | 0.0414 |
| 35 | 25.7559 | 25.7784 | 25.8008 | 0.0448 |
| 36 | 23.3077 | 23.3316 | 23.3555 | 0.0476 |
| 37 | 20.4703 | 20.4886 | 20.5069 | 0.0365 |
| 38 | 18.7730 | 18.7946 | 18.8165 | 0.0433 |
| 39 | 19.4873 | 19.5040 | 19.5208 | 0.0333 |
| 40 | 10.8384 | 10.8523 | 10.8663 | 0.0278 |

Span of observations = 0.0274 (m)

Iteration number = 2

Table 4.4

| Point Nr. | Interval-Adjustment: span (x) (m) | Classic Adjustment: span ($m_x$) (m) | span (x) / span ($m_x$) |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 1 | 0,0001 | 0,0000 | – |
| 2 | 0,0807 | 0,0200 | 4,04 |
| 3 | 0,0781 | 0,0222 | 3,52 |
| 4 | 0,0597 | 0,0184 | 3,24 |
| 5 | 0,0647 | 0,0198 | 3,27 |
| 6 | 0,0645 | 0,0220 | 2,93 |
| 7 | 0,0607 | 0,0248 | 2,45 |
| 8 | 0,0742 | 0,0252 | 2,94 |
| 9 | 0,0555 | 0,0286 | 1,94 |
| 10 | 0,0703 | 0,0280 | 2,51 |
| 11 | 0,0694 | 0,0270 | 2,57 |
| 12 | 0,0491 | 0,0248 | 1,98 |
| 13 | 0,0618 | 0,0270 | 2,22 |
| 14 | 0,0564 | 0,0278 | 2,03 |
| 15 | 0,0663 | 0,0292 | 2,27 |
| 16 | 0,0570 | 0,0288 | 1,98 |
| 17 | 0,0584 | 0,0290 | 2,01 |
| 18 | 0,0603 | 0,0312 | 1,93 |
| 19 | 0,0669 | 0,0342 | 1,96 |
| 20 | 0,0608 | 0,0310 | 1,96 |
| 21 | 0,0571 | 0,0304 | 1,88 |
| 22 | 0,0539 | 0,0308 | 1,75 |
| 23 | 0,0607 | 0,0320 | 1,90 |
| 24 | 0,0564 | 0,0318 | 1,77 |
| 25 | 0,0523 | 0,0324 | 1,61 |
| 26 | 0,0508 | 0,0340 | 1,49 |
| 27 | 0,0531 | 0,0330 | 1,61 |
| 28 | 0,0589 | 0,0338 | 1,74 |
| 29 | 0,0564 | 0,0338 | 1,67 |
| 30 | 0,0519 | 0,0334 | 1,55 |
| 31 | 0,0456 | 0,0356 | 1,28 |
| 32 | 0,0519 | 0,0356 | 1,46 |
| 33 | 0,0540 | 0,0370 | 1,46 |
| 34 | 0,0415 | 0,0344 | 1,21 |
| 35 | 0,0449 | 0,0346 | 1,30 |
| 36 | 0,0478 | 0,0342 | 1,40 |
| 37 | 0,0366 | 0,0352 | 1,04 |
| 38 | 0,0435 | 0,0364 | 1,20 |
| 39 | 0,0335 | 0,0344 | 0,97 |
| 40 | 0,0279 | 0,0396 | 0,70 |

Mean : 7,01

——— : Minimum value

==== : Maximum value

Table 4.5

Definition 4.2 : A real matrix $A = [a_{ik}]$ will be called M-Matrix exactly when

$a_{ik} \leqslant 0$ for all $i \neq k$ and one of the following conditions is fullfilled:

[i] A is not singular and $A^{-1} \geqslant 0$

[ii] The diagonal $D := [a_{ii}]$ of A is positive and the spectral radius $\rho[E - D^{-1} A] < 1$ .

[iii] There is a M-Matrix $B \leqslant A$.

[iv] All values of A have positive real part.

[v] From $A X \geqslant 0$ results that $X \geqslant 0$ for all vectors x.

It may be noticed that for interval equations of the form:
$[A][X] = [b]$, ..... [4.15

where each matrix $A \in [A]$, it can be proved that a M-Matrix is the optimum proper of the iteration solution.

The following thesis is valid:
Thesis 4.1 [BARTH[1974] ] /2/:
For a M-Matrix interval] [A], which can be divided into

[L] lower three corner matrix
[D] diagonal matrix and
[U] upper three-corner matrix

$[[A] = [L] + [D] + [U]]$

The total steps method [Jacobi] and the single step-method [Gauss-seidel], both methoc converge by

$[X] \subseteq [X^{[0]}]$

forwards the optimum locking up $[\hat{X}]$ of the interval cover of the system [4.15].

Total steps method:

$[X^{[m+1]}] := [[D]^{-1} [[b] - [[L] + [U]][X^{[m]}]] \cap [X^{[m]}]$ for m = 0,1, ...

Single steps method:

$[X^{[m+1]}] := [[D]^{-1} [[b] - [L][X^{[m+1]}] - [U][X^{[m]}]]] \cap [X^{[m]}]$ for m = 0,1, ...

Thesis 4.1 gives an example how can the numerical experience within the scope of the interval calculations be exact and provable. Based on these thoughts, the interval adjustment used the normal equation of the form.

$A^t P A [X] = [A^t PL]$ , ($A^t P A$ = M-Matrix) ..... [4.16]

as a start to obtain the optimum solution according to thesis 4.1. While the used program is carried out with machine-interval-arithmatic using approximation [see chap. 1.2.1], therefore, the solution can be only an upper cover. A better interval upper cover, consequently the interval cover , can be obtained with other type of interval arithmatic . Therefore, the resulting solution can be considered as " Optimum-arithmatic-upper-cover".

Table 4.6 presents the results.

The mean value differs maximum with 0,0009 m [point 5 & 15] in comparison with the mean value obtained from krawczyk method [tab. 4.4]. The reason for this difference is the average formation in the interation [chap. 4.3.2]. The mean value of table 4.6 differs only a small amount from that of table 4.4 [krawczyk-method] inspite of the 19 iteration steps compared with only 2 in [krawczyk]. The reason is the first interval-locking up $[X^{[0]}]$ and not the iteration-steps number.

Table 4.7 presents the result interval span, the mean-error-interval span and the ratio of the 1st to the 2nd.

The results discussion of this part can be summeri ed in the following:

The interval analytical adjustment of a unidimensional net using modified observations lead to a normal equation of the form (4.16) While the normal equation matrix fulfills the M-Matrix property, the optimum solution can be obtained by using the total steps-iteration method, which is based on the machine-interval-arithmatic and can be only an upper cover. The 1st solution locking up must be

$$[\, x^{[0]} \,] \supseteq [\, X \,]$$

as eg. in the krawczyk method.Under these conditions[chap.4.1],the best fit value of the unknown will be surely found within the results interval. The spans are in the average 1,9 times greater than the " mean error interval spans ".

From these results the following conclusion can be formulated: in a great net starting with certain number of points or certain range [which must not be reached always] the optimum locking up is smaller than the mean-error-interval inspite of approximation, and noticing that mean-error-interval will continuously increase according to the error propagation laws.
This range will be reached in the test example at point 39. The interval adjustment method presents a real alternative for error calculation in the adjustment of unidimensional nets. A final judgement of the span in the test example from the different interval method will be in the following chapter.

## 4.5 Comparison of Results:

Comparison of the results-span represents the best way to express the quality of the interval method adjustment methods. Figure 4.1 represents the spans of the results obtained by the different methods for the test example. This graphical representation shows that the interval adjustment of unidimensional nets using the M-matrix property gives the optimum solution locking which may be an alternative to the classic adjustment method.

On comparing the interval algorithm on the number of the iteration steps, it was found: .

- classic adjustment Tab. 3.3 7 sec
- Leftinverted Interval adjustment tab. 4.1 ~ 46 sec.
- Interval adjustment through krawczyk tab. 4.4 ~ 70 sec.
- Interval adjustment through M-Matrix property tab. 4.6 ~ 178 sec.

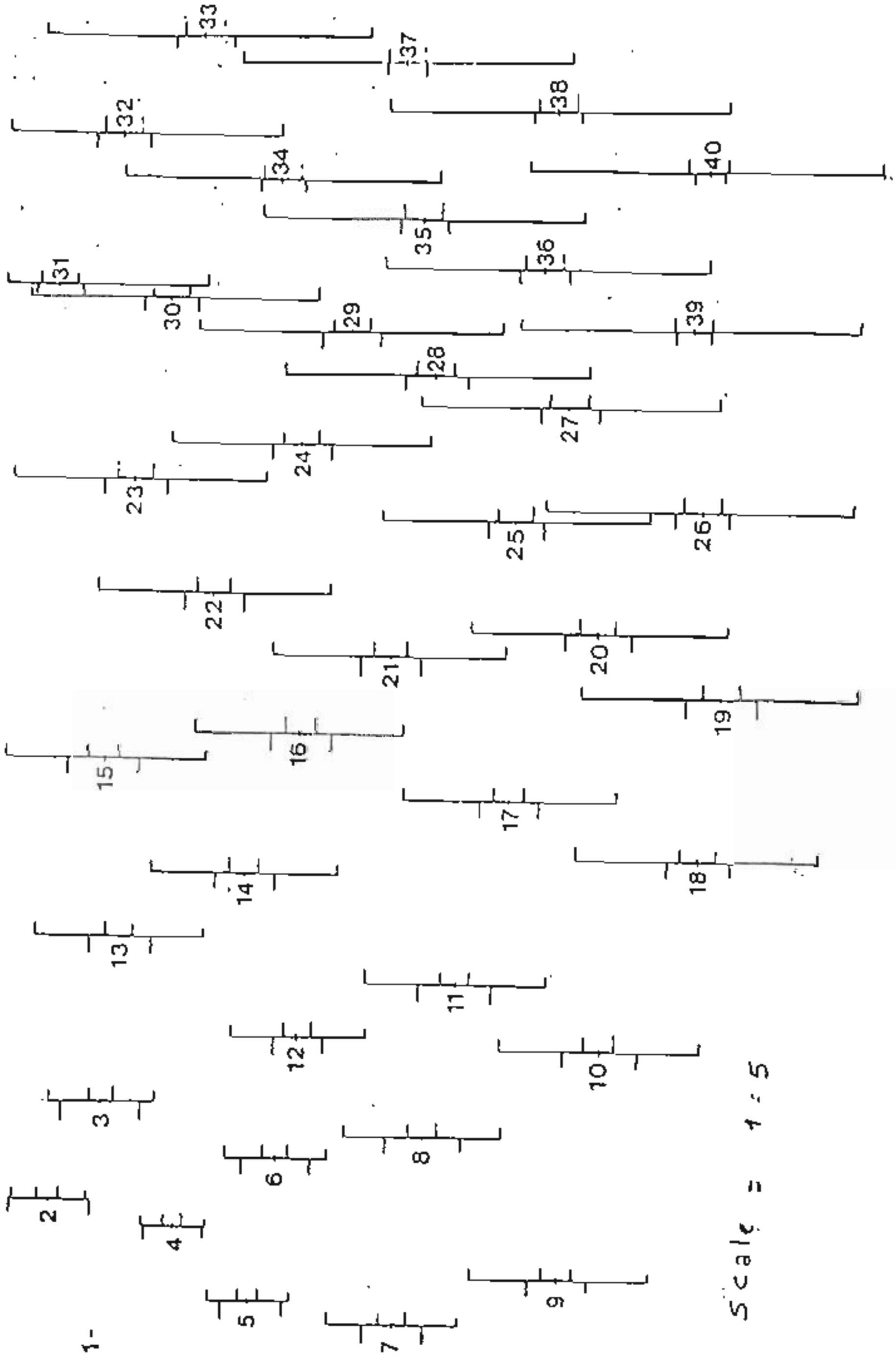



Fig. 4.1
Comparison of spans

mean-error-span

span using left-inverted Matrix

span using M-Matrix

# INTERVAL ADJUSTMENT
## ( M - MATRIX )

| Point Nr. | Lower Limit (m) | Mean (m) | Upper Limit (m) | Span (dx) (m) |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 |
| 1 | 9.3859 | 9.3860 | 9.3860 | 0.0000 |
| 2 | 7.5100 | 7.5478 | 7.5857 | 0.0754 |
| 3 | 10.0435 | 10.0016 | 10.1104 | 0.0747 |
| 4 | 8.1055 | 8.1336 | 8.1621 | 0.0565 |
| 5 | 15.8926 | 15.9232 | 15.9539 | 0.0612 |
| 6 | 26.0093 | 26.0404 | 26.0714 | 0.0619 |
| 7 | 17.9519 | 17.9812 | 18.0105 | 0.0585 |
| 8 | 24.8107 | 24.8459 | 24.8812 | 0.0703 |
| 9 | 18.4111 | 18.4379 | 18.4646 | 0.0533 |
| 10 | 16.2107 | 16.2449 | 16.2796 | 0.0681 |
| 11 | 26.8904 | 26.9245 | 26.9585 | 0.0679 |
| 12 | 28.0409 | 28.0645 | 28.0879 | 0.0469 |
| 13 | 15.0885 | 15.1186 | 15.1486 | 0.0599 |
| 14 | 20.6206 | 20.6481 | 20.6758 | 0.0551 |
| 15 | 23.5847 | 23.6176 | 23.6503 | 0.0654 |
| 16 | 30.0012 | 30.0292 | 30.0570 | 0.0557 |
| 17 | 28.4822 | 28.5110 | 28.5396 | 0.0573 |
| 18 | 20.1449 | 20.1747 | 20.2045 | 0.0594 |
| 19 | 21.9495 | 21.9826 | 22.0156 | 0.0659 |
| 20 | 24.8116 | 24.8421 | 24.8724 | 0.0607 |
| 21 | 40.0774 | 40.1060 | 40.1345 | 0.0569 |
| 22 | 31.5377 | 31.5643 | 31.5907 | 0.0529 |
| 23 | 25.3571 | 25.3874 | 25.4176 | 0.0603 |
| 24 | 35.4321 | 35.4601 | 35.4881 | 0.0558 |
| 25 | 31.5985 | 31.6242 | 31.6479 | 0.0513 |
| 26 | 25.6097 | 25.6349 | 25.6602 | 0.0503 |
| 27 | 27.0271 | 27.0533 | 27.0795 | 0.0522 |
| 28 | 30.0574 | 30.0866 | 30.1150 | 0.0582 |
| 29 | 29.6832 | 29.7112 | 29.7392 | 0.0559 |
| 30 | 27.4537 | 27.4793 | 27.5050 | 0.0511 |
| 31 | 24.2474 | 24.2701 | 24.2928 | 0.0452 |
| 32 | 19.8803 | 19.9060 | 19.9317 | 0.0513 |
| 33 | 15.4973 | 15.5241 | 15.5509 | 0.0535 |
| 34 | 22.1160 | 22.1367 | 22.1573 | 0.0411 |
| 35 | 25.7561 | 25.7785 | 25.8007 | 0.0444 |
| 36 | 23.3079 | 23.3316 | 23.3554 | 0.0474 |
| 37 | 20.4704 | 20.4886 | 20.5068 | 0.0363 |
| 38 | 18.7731 | 18.7946 | 18.8166 | 0.0433 |
| 39 | 19.4874 | 19.5041 | 19.5206 | 0.0332 |
| 40 | 10.8384 | 10.8523 | 10.8663 | 0.0278 |

Table 4.6

Span of observations = 0.0276

Iteration number = 19

| | Interval – Adjustment | Classic – Adjustment | |
|---|---|---|---|
| Point Nr. | span (x) (m) | span ($m_x$) (m) | span (x) / span ($m_x$) |
| 1 | 2 | 3 | 4 |
| 1 | 0,0001 | 0,0000 | – |
| 2 | 0,0757 ==== | 0,0200 | 3,79 ==== |
| 3 | 0,0749 | 0,0222 | 3,37 |
| 4 | 0,0566 | 0,0184 —— | 3,08 |
| 5 | 0,0613 | 0,0198 | 3,10 |
| 6 | 0,0621 | 0,0220 | 2,82 |
| 7 | 0,0586 | 0,0248 | 2,36 |
| 8 | 0,0705 | 0,0252 | 2,80 |
| 9 | 0,0535 | 0,0286 | 1,87 |
| 10 | 0,0683 | 0,0280 | 2,44 |
| 11 | 0,0681 | 0,0270 | 2,52 |
| 12 | 0,0470 | 0,0248 | 1,90 |
| 13 | 0,0601 | 0,0270 | 2,23 |
| 14 | 0,0552 | 0,0278 | 2,00 |
| 15 | 0,0656 | 0,0292 | 2,25 |
| 16 | 0,0558 | 0,0288 | 1,94 |
| 17 | 0,0574 | 0,0290 | 1,98 |
| 18 | 0,0596 | 0,0312 | 1,91 |
| 19 | 0,0661 | 0,0342 | 1,93 |
| 20 | 0,0608 | 0,0310 | 1,96 |
| 21 | 0,0571 | 0,0304 | 1,88 |
| 22 | 0,0530 | 0,0308 | 1,72 |
| 23 | 0,0605 | 0,0320 | 1,89 |
| 24 | 0,0560 | 0,0318 | 1,76 |
| 25 | 0,0514 | 0,0324 | 1,59 |
| 26 | 0,0505 | 0,0340 | 1,49 |
| 27 | 0,0524 | 0,0330 | 1,59 |
| 28 | 0,0584 | 0,0338 | 1,73 |
| 29 | 0,0560 | 0,0338 | 1,66 |
| 30 | 0,0513 | 0,0334 | 1,54 |
| 31 | 0,0454 | 0,0356 | 1,28 |
| 32 | 0,0514 | 0,0356 | 1,44 |
| 33 | 0,0536 | 0,0370 | 1,45 |
| 34 | 0,0413 | 0,0344 | 1,20 |
| 35 | 0,0446 | 0,0346 | 1,29 |
| 36 | 0,0475 | 0,0342 | 1,39 |
| 37 | 0,0364 | 0,0352 | 1,03 |
| 38 | 0,0435 | 0,0364 | 1,20 |
| 39 | 0,0334 | 0,0344 | 0,97 |
| 40 | 0,0279 —— | 0,0396 ==== | 0,70 —— |

Mean = 1,92

—— = Minimum Value

==== = Maximum Value

Table 4.7

## INTERVAL ADJUSTMENT

## ( KRAWCZYK )

| Point Nr. | Lower limit (m) | Adjusted value (m) | Upper Limit (m) | Span ($d_x$) (m) |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 |
| 1 | 8.6801 | 9.3860 | 9.6939 | 1.0117 |
| 2 | 7.0505 | 7.5470 | 8.0381 | 0.9705 |
| 3 | 9.6551 | 10.0770 | 10.5039 | 0.8477 |
| 4 | 7.6536 | 8.1341 | 8.6156 | 0.9609 |
| 5 | 15.4430 | 15.9225 | 16.4021 | 0.9570 |
| 6 | 25.6065 | 26.0363 | 26.4650 | 0.8574 |
| 7 | 17.5614 | 17.9765 | 18.3906 | 0.8281 |
| 8 | 24.4689 | 24.8386 | 25.1873 | 0.6973 |
| 9 | 18.0644 | 18.4321 | 18.7789 | 0.6934 |
| 10 | 16.0137 | 16.2417 | 16.4703 | 0.4561 |
| 11 | 26.6776 | 26.7199 | 27.1597 | 0.4795 |
| 12 | 27.7180 | 28.0598 | 28.4026 | 0.6836 |
| 13 | 14.8181 | 15.1140 | 15.4090 | 0.5890 |
| 14 | 20.4062 | 20.6421 | 20.8785 | 0.4717 |
| 15 | 23.4687 | 23.6074 | 23.7505 | 0.2812 |
| 16 | 29.8835 | 30.0227 | 30.1624 | 0.2783 |
| 17 | 28.3750 | 28.5679 | 28.6403 | 0.2646 |
| 18 | 20.1384 | 20.1795 | 20.3212 | 0.2822 |
| 19 | 21.8108 | 21.9875 | 22.1670 | 0.3564 |
| 20 | 24.6857 | 24.8459 | 25.0056 | 0.3193 |
| 21 | 39.9808 | 40.1019 | 40.2233 | 0.2422 |
| 22 | 31.4268 | 31.5552 | 31.6832 | 0.2559 |
| 23 | 25.2035 | 25.3816 | 25.5600 | 0.3564 |
| 24 | 35.2607 | 35.4551 | 35.6500 | 0.3807 |
| 25 | 31.4104 | 31.6231 | 31.8282 | 0.4092 |
| 26 | 25.4058 | 25.6302 | 25.8702 | 0.4639 |
| 27 | 26.7902 | 27.0559 | 27.3206 | 0.5293 |
| 28 | 29.7893 | 30.0891 | 30.3899 | 0.5996 |
| 29 | 29.4104 | 29.7093 | 30.0091 | 0.5977 |
| 30 | 27.2025 | 27.4700 | 27.7544 | 0.5508 |
| 31 | 23.9991 | 24.2667 | 24.5353 | 0.5352 |
| 32 | 19.5610 | 19.9061 | 20.2294 | 0.6465 |
| 33 | 15.1702 | 15.5267 | 15.8822 | 0.7109 |
| 34 | 21.8006 | 22.1405 | 22.4794 | 0.6777 |
| 35 | 25.4440 | 25.7829 | 26.1208 | 0.6758 |
| 36 | 22.9970 | 23.3330 | 23.6680 | 0.6699 |
| 37 | 20.1346 | 20.4920 | 20.8485 | 0.7129 |
| 38 | 18.4404 | 18.7969 | 19.1524 | 0.7109 |
| 39 | 19.1869 | 19.5083 | 19.8286 | 0.6406 |
| 40 | 10.5013 | 10.8569 | 11.2134 | 0.7109 |

Table 4.8

Span of observations = 0.0274 (m)

Number of iterations = 2

The iteration steps in the step iteration is 19, in the krawczyk method is 2. Noticing that these expences are possible, so those both critaria will be considered of second importance.

## 4.6 Short Note to the Adjustment of a free Unidimensional Net, also to Interval adjustment of Observation with Limited Validity :

In case of adjustment valid under certain conditions the observations must be adjusted using the correlation-equation. A solution through the interval-adjustment under certain conditions will not be suitable.

If a unidimensional net will be free adjustment [test example Level net], the normal equation matrix based on a degree of freedom [therefore singular] must be transformed to a regular matrix. This can be achieved by increasing a column and a line in the normal equation matrix [GOTTHARQT [1978] /6 /]. To see the reflection of this change on the results the test example has been free adjusted and starting from this normal equation an interval adjustment using the krawczyk-method has been carried out.

Interval adjustment through total steps and single steps iteration method is impossible because $D^{-1}$ $(d_n + 1, n + 1 = 0)$ does not exist [see thesis 4.1]

The free adjustment has been carried out using the computer program. The net was then transformed on point against the modified adjustment [see tab. 3.3]. The interval-adjustment was then carried out using krawczyk method.

The results are summerized in table 4.8. Comparing these results with those in table 4.4 paying attention to the unrealistic great span, the results of the modified adjustment through the krawczyk-method do not need any comment to explain their quality. It is clear that the sign condition of the normal equation matrix after simplified observations and not the krawczyk-method which leads to these good solutions locking up.

## SUMMARY :

After introduction of some definitions and thesis from the interval-calculation theory followed by a short discussion of the computer interval language.

It has been shown that on introduction of observed intervals every normal equation is transferred to the modified interval equation, for which the definition and thesis of the interval analysis are valid. After that, different methods of the interval analytical adjustment have been introduced. Their quality have been tested by comparing the results obtained for a testing example [unidimensional Level net]. The left invers Matrix of the reduced-equation matrix of the independent interval gave unrealisetic result span. If we pay attention to the sign distribution of the normal-equation matrix we can obtain an optimum solution through the M-Matrix properties of the normal-equation matrix. The solution obtained will be only an upper cover due to the used machine-interval-mathematics. This will be obtained through the Interval-total steps, using a secured first solution of krawczyk algorithmus.

This interval adjustment method presents an alternative to the error propagation, based on the optimum property of the solution and the obtained results. All programs and examples of this research have been set and calculated in computer center of Elmansoura University.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# APPENDIX (A)

A. Program of classic Adjustment for unidimensional level net

1- Parameter - list

2- Description the flow chart program

1- Parameter - list:

INPUT ;

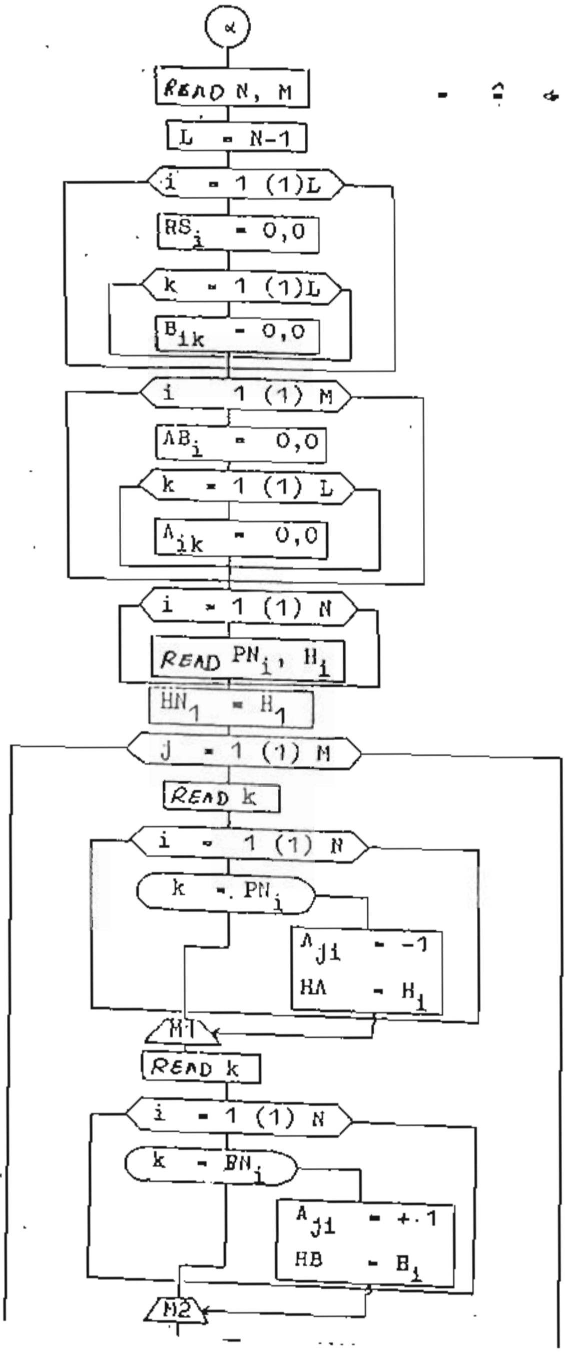
| Name | Type | |
|---|---|---|
| N | integer | = Total number of net - points |
| M | integer | = Total number of observations |
| PN | integer array | = Vector of points number |
| H | real array | = Vector of approx. levels |
| K | integer | = Point number for given observations |
| DH | real array | = Observations Vector |
| P | real array | = Weight matrix |

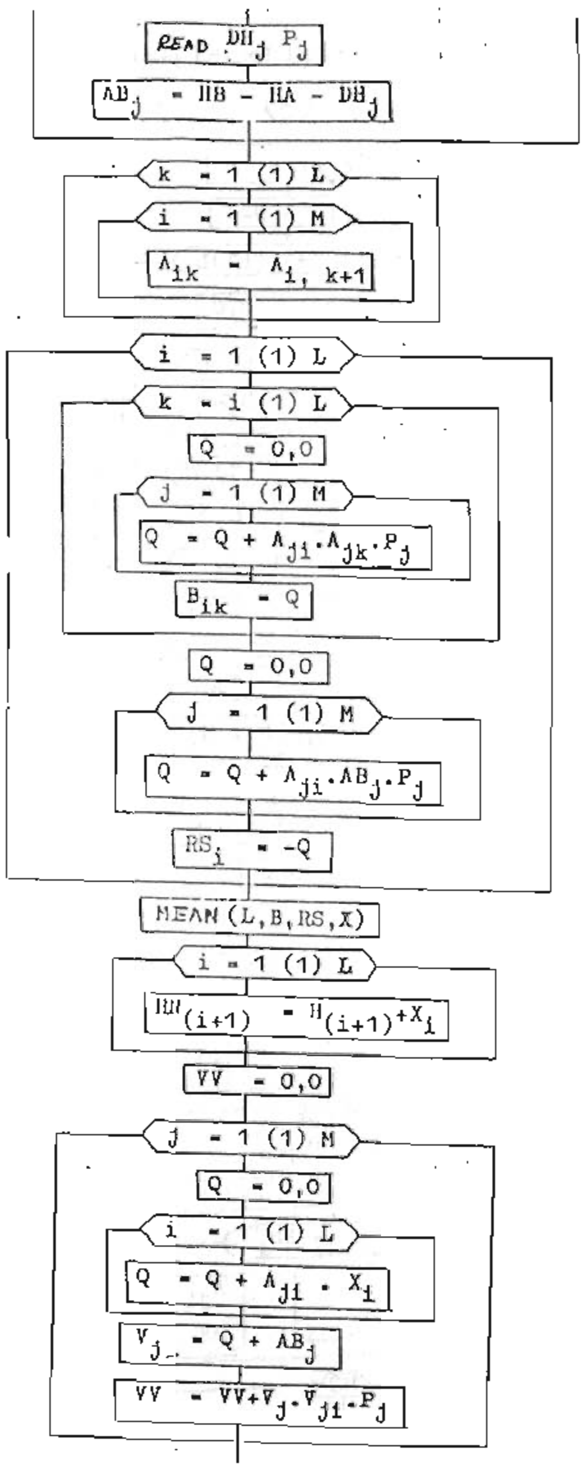
OUTPUT:

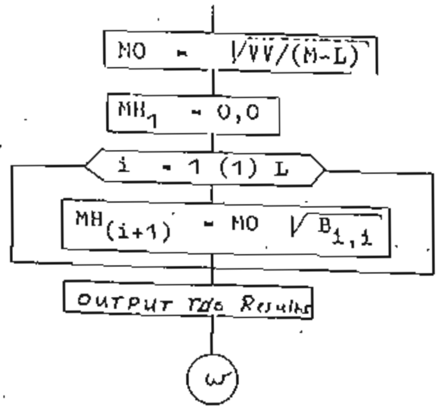
| Name | Type | |
|---|---|---|
| PN | | as given before |
| H | | as given before |
| HN | real array | = Vector of the adjusted levels |
| MH | real array | = Vector of the mean square error for the adjusted levels |
| $M_o$ | real | =mean square error for weifht unity |

2. Description the Flow Chart

Description of the flow chart for the program.




READ DH_j P_j
AB_j = HB - HA - DB_j
k = 1 (1) L
i = 1 (1) M
A_ik = A_i, k+1
i = 1 (1) L
k = i (1) L
Q = 0,0
j = 1 (1) M
Q = Q + A_ji.A_jk.P_j
B_ik = Q
Q = 0,0
j = 1 (1) M
Q = Q + A_ji.AB_j.P_j
RS_i = -Q
MEAN (L,B,RS,X)
i = 1 (1) L
HH_(i+1) = H_(i+1)+X_i
VV = 0,0
j = 1 (1) M
Q = 0,0
i = 1 (1) L
Q = Q + A_ji . X_i
V_j = Q + AB_j
VV = VV+V_j.V_ji.P_j


MO = √VV/(M-L)
MH1 = 0,0
i = 1 (1) L
MH(i+1) = MO √B i,i
OUTPUT The Results