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On an Integral Equation Method for a Linearized Version of the Navier-Stokes Equation.

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ON AN INTEGRAL EQUATION METHOD FOR A LINEARIZED VERSION OF THE NAVIER-STOKES EQUATION

حول طريقة معادلات تكاملية لحل صورة خطية من معادلة نافييه ـ ستوكس

By

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ملخص: يتناول هذا البحث طريقة المعادلات التكاملية المختزلة النظامية و التى سبق اقتراحها لمعالجة مسائل السريان الزاحف ، هذه الطريقة لم تكن تستلزم اكثر من حل معادلة مقياسية على السطح ، فى هذا البحث اقترح تحويل جديد للمتغيرات يسمح بمد مجال تطبيق الطريقة لتناول المسائل التى يكون فيها الحمل غير مهملا وذلك بعد تحويلها لصورة خطية مناسبة ، تم تطبيق هذه الطريقة الجديدة على منطقة الدخول الهيدروديناميكى فى مسلك مستطيل المقطع بطول لا نهائى .

Abstract : The regular reductive integral equations method, developed earlier to solve creeping flow problems, is considered. This method required merely the solution of a scalar integral equation on the boundary of the domain. Through a judicious linearization and transformation of variables, this method is extended to include cases with non-negligible convective term. This new technique is applied to the hydrodynamic entrance zone in a rectangular duct of infinite length.

1-INTRODUCTION

Fluid mechanics and heat transfer problems can be formulated in either a differential or an integral form. Differential formulations, which involve continuity, Navier-Stokes and eventually the energy equations, are, up till now, the most widely used. However, integral formulations, which require the transformation of the partial differential system of equations into an integral system, are more promising and constitute a rapidly developing field of computational thermofluids. This has two main reasons :

1- Approximate methods (like weighted residuals, finite elements, perturbations,.. etc.), which are extensively used in differential formulations, are also applicable to integral formulations, and perform generally better in the later case. The higher convergence rate encountered is attributable to the elimination of delicate operations inherent in differential formulations like numeric differentiation, or differentiating an infinite series. The analysis required to eliminate these operations ensures the embodiment of the correct qualitative behavior in the solution.
2- The problem to be solved can be greatly reduced when recast in an integral form. In fact, the number of independent variables (and sometimes the dependent variables also) are usually reduced, since integral equations are generally defined on the boundary of the domain. This may open the realm of 3-D time dependent problems to systems with modest computational resources.

These decisive advantages are not without a counterpart. In fact, an extensive preliminary analysis should be performed in order to find the equivalent integral formulation, which has so far been done for only few cases.

In this work, after a brief overview of integral equation methods (in section 2 where more light is shed on the regular reductive method), a new transformation will be proposed in section 3 in order to extend also vanish at the duct walls ($x=0$ or a, and $y=0$ or b), and acquire the fully developed regime at exit (at infinity). At the fully developed regime, the velocity and pressure gradient have only the axial component $w_{fd}(x,y)$ and $-C$ respectively (where C is a constant). This leads to the following system :

$$(1/RE)\, \nabla^2 V = (V \cdot \nabla) V + \nabla P \qquad 4.1\ a$$
$$\nabla \cdot V = 0 \qquad 4.1\ b$$
$$n \cdot V = 0 \quad \text{for } x=0,\ x=a,\ y=0,\ \text{or } y=b \qquad 4.1\ c$$
$$n \cdot V = -1 \quad \text{for } z=0 \qquad 4.1\ d$$
$$n \cdot V = w_{fd} \quad \text{for } z \to \infty \qquad 4.1\ e$$
$$n * V = 0 \quad \text{on } \partial\Omega \qquad 4.1\ f$$

The above system has been made nondimensional using the hydrodynamic mean depth as the characteristic length and the amplitude of the constant velocity at inlet as a characteristic velocity. The fully developed velocity and pressure gradient are obtainable from:

$$\nabla^2 w_{fd}(x,y) = -\, RE\ C \qquad 4.2\ a$$
$$w_{fd}(x,y) = 0 \quad \text{for } x=0,\ x=a,\ y=0 \ \text{or } y=b \qquad 4.2\ b$$
$$\int w_{fd}\ dx\ dy = a\ b \qquad 4.2\ c$$

by separation of variables, which yields :

$$w_{fd} = RE\ C\ (4/a)\ \Sigma_{n\ odd} \sin(n\pi x/a)\ (n\pi/a)^{-3}\ f_n(y) \qquad 4.3\ a$$
$$f_n(y) = 1 - [\sinh(n\pi y/b) + \sinh(n\pi(b-y)/a)]/\sinh(n\pi b/a) \qquad 4.3\ b$$
$$RE\ C = ab/\{(8/a)\ \Sigma_{n\ odd}(n\pi/a)^{-4}\ b[1-\tanh(n\pi b/(2a))/(n\pi b/(2a))]\} \qquad 4.3\ c$$

Applying the transformation exposed in section 3, taking U as the inlet velocity, we get for the first iteration (i.e. neglecting the minor nonlinear part of the convective term) :

$$(\nabla^2 + K^2)\ v = (RE/e)\ \nabla P \qquad 4.4\ a$$
$$\nabla^2 P = 0 \qquad 4.4\ b$$
$$\nabla \cdot v = -\,(RE/2)\ z.v \quad \text{(z being the unit vector in the z direction)}$$
$$= RE/(2\ e) \quad \text{for } x=0,\ x=a,\ y=0 \ \text{or } y=b \qquad 4.4\ c$$
$$= 0 \quad \text{for } z=0 \qquad 4.4\ d$$
$$= -\,(RE/2)\ (w_{fd}-1) \quad \text{for } z \to \infty \qquad 4.4\ e$$
$$n \cdot v = 0 \quad \text{for } z=0,\ x=0,\ x=a,\ y=0 \ \text{or } y=b \qquad 4.4\ f$$
$$= w_{fd} - 1 \quad \text{for } z \to \infty \qquad 4.4\ g$$
$$n * v = (z * n)/e \quad \text{for } x=0,\ x=a,\ y=0 \ \text{or } y=b \qquad 4.4\ h$$
$$= 0 \quad \text{for } z=0,\ z \to \infty \qquad 4.4\ i$$

The following complete system at the boundary, expresses the pressure in terms of the unknown pressure constants A_{jk}, B_{ik}, C_{ij} :

$$P = \Sigma_{jk}\ A_{jk}\ Y_j(y)\ k\ \exp(-k\ z) - C\ z \quad \text{for } x=0 \text{ or } a \qquad 4.5\ a$$
$$= \Sigma_{ik}\ B_{ik}\ X_i(x)\ k\ \exp(-k\ z) - C\ z \quad \text{for } y=0 \text{ or } b \qquad 4.5\ b$$
$$= \Sigma_{ij}\ C_{ij}\ X_i(x)\ Y_j(y) \quad \text{for } z=0 \qquad 4.5\ c$$
$$X_i(x) = \sqrt{(2/a)}\ \sin(\ (2i-1)\pi x/a\ ) \qquad 4.5\ d$$
$$Y_j(y) = \sqrt{(2/b)}\ \sin(\ (2j-1)\pi y/b\ ) \qquad 4.5\ e$$

Solving 4.4 b, taking into account boundary conditions 4.5 and the fact that P tends to $-C\ z$ when z tends to infinity we get:

$$P = -\,C\ z + \Sigma_k\{\Sigma_j A_{jk}\ Y_j\ k\ e^{-kz}\ [\sinh(k_j'x) + \sinh(k_j'(a-x))]/\sinh(k_j'a)$$
$$\Sigma_i B_{ik}\ X_i\ k\ e^{-kz}\ [\sinh(k_i''y) + \sinh(k_i''(b-y))]/\sinh(k_i''b)\}$$
$$+\Sigma_{ij} X_i Y_j\{e^{-\alpha_{ij}z}\ (C_{ij} - \Sigma_k(\frac{A_{jk}\ k\ (2i-1)\pi\ 2\sqrt{(2/a)}}{a(\alpha_{ij}^2 - k^2)} + \frac{B_{ik}\ k\ (2j-1)\pi\ 2\sqrt{(2/b)}}{b(\alpha_{ij}^2 - k^2)}\ )))\} \qquad 4.6\ a$$

the domain of applicability of the considered method. An application of the extended version will be given in section 4, followed by a conclusion in section 5.

2- OVERVIEW ON INTEGRAL EQUATIONS METHODS

2-1 GENERAL

It is not an evident, nor an easy, task to give a unified presentation and classification of such a rapidly developing field of computational thermofluids. An attempt towards the construction of this presentation will be given here. Consider the following general problem:

$$T[U] = g \quad \text{in} \quad \Omega \qquad 2.1\text{ a}$$
$$C(U) = u_s \quad \text{on} \quad \delta\Omega \qquad 2.1\text{ b}$$

where $T[.]$ is a differential operator, U the unknown function, $C[.]$ an operator expressing boundary conditions, Ω and $\delta\Omega$ are respectively the domain and its boundary, and g and u_s are known functions.

Assume that the solution of a qualitatively similar problem is known:

$$T'(U] = g' \quad \text{in} \quad \Omega \qquad 2.2\text{ a}$$
$$C'[U] = u_s' \quad \text{on} \quad \delta\Omega \qquad 2.2\text{ b}$$

By qualitatively similar we mean that T and T' are of the same order and type (ex. both elliptic or parabolic,..) and that $\Omega < \Omega'$. The general solution can be expressed using the Green's function $G(r,r')$ of the auxiliary problem in the following form :

$$U = G_{vv}[g'] + G_{vs}[u_s'] \qquad 2.3$$

where $G[.]$ is an integral operator having the Green's function as a kernel, the first subscript represents the domain of definition of the image (v for volume and s for surface), the second that of the range :

$$G_{vv}[g'] = \int_{r'\varepsilon\Omega} G(r,r')\, g'(r')\, dv' \qquad 2.4\text{ a}$$

$$G_{vs}[u_s'] = \int_{r'\varepsilon\delta\Omega} G'(r,r')\, u_s'(r')\, da' \qquad 2.4\text{ b}$$

where $G'(r,r')$ is obtained from $G(r,r')$ using standard procedures (cf. MORSE & FESHBACH (1953). Note that $T'[G_{vs}[.]]=0$, $C'[G_{vv}[.]]=0$.

Defining $T_d[.]$ (the difference operator) as:

$$T_d[.] = T'[.] - T[.] \qquad 2.5$$

and assigning to g' the value $g + T_d[U]$, equation 2.3 takes the form :

$$U = G_{vv}[T_d[U] + g] + G_{vs}[u_s'] \quad \text{in} \quad \Omega \qquad 2.6$$

It is clear that the above expression of U satisfies 2.1 a . Substituting in 2.1 b we get :

$$C[G_{vv}[T_d(U) + g] + G_{vs}[u_s']]=u_s \quad \text{on} \quad \delta\Omega \qquad 2.7$$

Equations 2.6 and 2.7 constitute a system of integral equations in the unknowns U , u_s' . Generally, we have to solve only one of these equations. If $T'[.] = T[.]$ then $T_d[.] = 0$, and we have to solve 2.7

only to get u_s', then substitute in 2.6 to get U. If C'[.] = C[.] then we can put $u_s'=u_s$ to make 2.7 trivial, and solve 2.6 to get directly U.

Integral equation methods can thus be classified into non reductive, when we have to solve 2.6 in the whole domain to get U, and reductive, when we have to solve 2.7 on $\delta\Omega$ only to get u_s' and hence U.

2-2 NON REDUCTIVE METHODS

As has been mentioned above, in this case equation 2.6 should be solved in Ω to get u. The introduction of the Green function of a resembling problem means that the problem is partially resolved analytically, which results in a good qualitative description of the solution behavior. Using standard procedures (like discretization or series expansion), we get an algebraic system whose matrix is not multidiagonal but is diagonally dominant. This disadvantage can be overcome by regionalizing the method in order to have a multidiagonal matrix by block (since each region is directly related to adjacent regions only). These methods are called Local Green's Function Methods. In a comparative review (DORNING 1981) has observed a reduction of up to 1000 times in computer time compared to finite difference methods.

2-3 SINGULAR REDUCTIVE METHODS

Reductive methods, where we have to solve 2.7 on $\delta\Omega$ only, can be subclassified into singular and regular (discussed in the next paragraph).

In singular methods, the domain Ω' is infinite allowing the possibility to find relatively easily the inverse of T[.]. Hence, T'[.] is chosen identical to T[.], and u_s' is fixed such as to guarantee regularity at infinity. The source term g' is identical to g inside Ω, vanishes outside Ω and assumes a distribution of singularities on $\delta\Omega$:

$$g' = g + g_s\ \delta(r - r_s) \qquad r_s \in \delta\Omega \qquad 2.8$$

where $\delta(.)$ is the Dirac distribution. This makes it easy to integrate :

$$G_{vv}\ [\ g_s\ \delta(r-r_s)\ ] = G_{vs}'\ [\ g_s\ ] \qquad 2.9$$

which is an integral equation on $\delta\Omega$ in the unknown function g_s. Once solved, U can be obtained easily from 2.6.

HESS and SMITH (1966) studied extensively the harmonic equation for potential flows, using this method. For viscous incompressible flows with two velocity components, it is possible to derive a biharmonic equation for the stream function and use this class of methods. This has been done by GLUCKMAN et. al. (1972) for axisymmetric flows and by COLEMAN (1981) for 2-D flows. Finally, YOUNGREN and ACRIVOS (1975) have derived a singular reductive method for Stokes' equation for 3-D exterior flows involving an unknown vector function on the boundaries.

2-4 REGULAR REDUCTIVE METHOD

Since the proposed method belongs to this category, a more detailed presentation will be given. In this method, Ω' coincides with Ω, but both the operators (T , T') and the boundary conditions (C , C') are in general different. The differences are chosen such as to reduce the following term by integration by parts :

$$G_{vv}\ [\ T_d\ [\ U\ ]\ ] = G_{vs}''[\ U\ ] \qquad 2.10$$

where the RHS represents the boundary term, whereas the other term resulting from integration by parts (the volume integral) vanish Hence by substituting in 2.7 we get an integral equation on the surfa in the unknown U (we can assign any desired value to u '). Once solv we can substitute in 2.6 to get U in the whole domain.

To be specific, let us consider a simple example which is linear hydrodynamic problem. To solve it we assume that the Green function of the vector Laplace equation is known in the considered geometries (see Appendix 1). The problem is governed by the continuity and Stokes' equations :

$$\nabla^2 \mathbf{V} = RE \ \nabla P \qquad 2.11\ a$$
$$\nabla \cdot \mathbf{V} = 0 \qquad 2.11\ b$$
$$\mathbf{n} \cdot \mathbf{V} = U_n \quad \text{on} \quad \delta\Omega \qquad 2.11\ c$$
$$\mathbf{n} * \mathbf{V} = \mathbf{U}_t \quad \text{on} \quad \delta\Omega \qquad 2.11\ d$$

where V is the velocity, P the pressure, n the unit outward normal, RE the Reynolds number, U_n and U_t are known functions, and the dot (.) and the asterisk (*) represent respectively the scalar and vector products (boldface characters represent vector functions or operators).

This problem has two unknown functions V and P and two partial differential equations. But the pressure equation is not explicit. Therefore, let us reconstruct it by applying the divergence to 2.11 and use 2.11 b to get :

$$\nabla^2 P = 0 \qquad 2.12$$

This could replace 2.11b if a new boundary condition were added, since it has a higher order. In fact, if we apply the divergence to 2.11 a and use 2.12 we find :

$$\nabla^2 (\nabla \cdot \mathbf{V}) = 0$$

It is evident that the missing boundary condition is simply :

$$\nabla \cdot \mathbf{V} = 0 \quad \text{on} \quad \delta\Omega$$

To sum up, the new system takes the form :

$$\nabla^2 \mathbf{V} = \nabla P \quad \text{in} \quad \Omega \qquad 2.13\ a$$
$$\nabla^2 P = 0 \quad \text{in} \quad \Omega \qquad 2.13\ b$$
$$\nabla \cdot \mathbf{V} = 0 \quad \text{on} \quad \delta\Omega \qquad 2.13\ c$$
$$\mathbf{n} \cdot \mathbf{V} = U_n \quad \text{on} \quad \delta\Omega \qquad 2.13\ d$$
$$\mathbf{n} * \mathbf{V} = \mathbf{U}_t \quad \text{on} \quad \delta\Omega \qquad 2.13\ e$$

Now let us invert the vector Laplacian operator in 2.13 a using boundary conditions 2.13 c,e :

$$\mathbf{V} = \mathbf{G} [\nabla P] - \mathbf{S} \qquad 2.14$$

where S is a known term that depends on U_t, and G(.) the green's tensor of the vector Laplacian operator corresponding to conditions 2.13 c,e. By applying the last condition 2.13 d we obtain the integral equation :

$$\mathbf{n} \cdot \mathbf{G} [\nabla P] = \mathbf{n} \cdot \mathbf{S} + U_n \quad \text{on} \quad \delta\Omega \qquad 2.15$$

SABRY (1984) has formally proved using 2.13 b that the integral in the LHS of 2.15 involves the values of P on the boundaries only. Another simpler approach, from the practical point of view, is to solve formally 2.13 b to get P in Ω in terms of the yet unknown values of the pressure on δΩ (P_s):

$$P = G [P_S] \qquad 2.16$$

Hence by substituting in 2.15 we get the sought for equation on $\delta\Omega$:

$$n \;.\; G [\nabla G [P_S]] = n \;.\; S \;+ U_n \quad \text{on} \;\; \delta\Omega \qquad 2.17$$

Once solved, we can substitute in 2.16 to get P and hence in 2.14 to get V.

The advantages of this approach include the use of the Green's function of a similar problem which ensures a high convergence rate (since the problem is partially resolved analytically). In addition, the reduction of the number of independent variables (solution on $\delta\Omega$) and dependent variables (only one scalar unknown function P_S) cause a dramatic decrease in computing effort. The only limits are the geometries in which this method can be applied (this can be relaxed on the expense of some of the above advantages cf. SABRY 1984), and the nature of the differential equation to be solved. In fact, the above exposed method is suitable for the Stokes' equation only (creeping flow). In this work, we propose to relax this restriction as will be shown in the next section.

3 - THE PROPOSED METHOD

In this section a new transformation is proposed which is applicable for cases where a constant average velocity U can be defined, such as flows around moving bodies and internal flows in ducts of constant cross-section.

In the first step, let us decompose the velocity field into a known constant value U and an unknown variable velocity u :

$$V = U + u \qquad 3.1$$

By substituting in the nondimentionalized Navier-Stokes' and continuity equations :

$$(1/RE) \;\nabla^2 V - (V \;.\; \nabla) V = \nabla P + S \qquad 3.2\,a$$

$$\nabla \;.\; V = 0 \qquad 3.2\,b$$

we get :

$$(1/RE) \;\nabla^2 u - (U \;.\; \nabla) v = \nabla P + S' \qquad 3.3\,a$$

$$\nabla \;.\; u = 0 \qquad 3.3\,b$$

where $\quad S' = S + (u \;.\; \nabla) u \qquad 3.3\,c$

In this step the nonlinear term has been decomposed into a major linear part (the second term in the LHS of 3.3 a) and a minor nonlinear part (the second term in the RHS of 3.3 c). In many cases this last term can be safely neglected, at least in the first approximation, to yield a linearized version of the Navier-Stokes' equation which is less restrictive than the Stokes' equation (2.13 a).

The second transformation consists of the substitution :

$$u = e\, v \qquad 3.4\,a$$

where $\quad e = \exp (a\, U \;.\; x) \qquad 3.4\,b$

and a is a constant to be determined shortly below. From the definition of the Laplacian operator, and using vector identities and 3.3 b, w have:

$$\nabla^2 u = \nabla (\nabla . u) - \nabla * (\nabla * u)$$

$$= 0 - \nabla * (\nabla * (e v))$$

$$= - \nabla * (e (\nabla * v) + (\nabla e) * v)$$

$$= - e \nabla * (\nabla * v) - (\nabla e) * (\nabla * v) - \nabla * ((\nabla e) * v)$$

The last term in the RHS can be decomposed into :

$$- \nabla * ((\nabla e) * v) = 2 [(\nabla e) . \nabla] v - \nabla[(\nabla e) . v] - (\nabla e)(\nabla.v) + v \nabla^2 e + (\nabla e) * (\nabla * v)$$

$$= 2 [(\nabla e) . \nabla] v - \nabla[(\nabla e) . v] - (\nabla[e (\nabla.v)] + e \nabla(\nabla.v) + v \nabla^2 e + (\nabla e) * (\nabla * v)$$

Since $\nabla e = e U$, we receive :

$$\nabla^2 u = e (\nabla^2 v + 2 a (U.\nabla)v + (aU)^2 v) \qquad 3.5 a$$

Also, the term (U.V) u becomes after the substitution of 3.4 a :

$$(U.\nabla) u = (U.\nabla) (e v)$$

$$= \{ \nabla(e U.v)-\nabla*(e U*v)+U(\nabla.(ev))-U*[\nabla*(e v)]\}/2$$

$$= e(U.\nabla)v + a e [(U.v)U - U*(U*v)]$$

$$= e \{ (U.V) v + a U^2 v \} \qquad 3.5 b$$

also, $\nabla.u = e (\nabla.v) + v (\nabla e) = 0$

hence $\nabla . v = -a U.v$ 3.5 c

Substituting of 3.5 in 3.3 we obtain :

$$(e/RE) \{\nabla^2 v + (2 a - RE) (U.\nabla) v + ((aU)^2 - a RE U^2) v \}$$

$$= \nabla P + S' \qquad 3.6$$

Now if we take a=RE/2 , the linearized part of the convective term (the second term in the LHS of 3.6) will vanish giving finally :

$$(\nabla^2 + K^2) v = (RE/e) (\nabla P + S') \qquad 3.7 a$$

$$\nabla . v = -(RE/2) (U.v) \qquad 3.7 b$$

where $K^2 = -(RE/2)^2$ 3.7 c

This is a vector Helmholtz equation whose Green's function can b easily obtained using standard procedures (see Appendix 1)

4 - APPLICATIONS

As an example, we have considered the problem of finding the 3- flow field in the entrance region of a rectangular duct of infinit length. The flow enters at a constant axial velocity (in the direction) and without any tangential component. The velocity sh where $\alpha_{ij} = \sqrt{((2i-1)\pi/a)^2 + ((2j-1)\pi/b)^2}$ 4.6 b

$k_j' = \sqrt{((2j-1)\pi/b)^2 - k^2}$ 4.6 c

$k_i'' = \sqrt{((2i-1)\pi/a)^2 - k^2}$ 4.6 d

Substituting in 4.4 a and solving formally taking into account boundary conditions 4.4 c,d,h and i (the condition for $z \to \infty$ is automatically taken into account due to the pressure form adopted), we obtain an expression for v (the Green's function for v is given in the appendix) in terms of the pressure constants. Finally, applying the last boundary condition (4.4 f) on this expression, and integrating over the surface (using as weights the functions over which the pressure is expanded), we get an algebraic system in the unknown pressure constants:

$$\Sigma_k \; A_{jk}\, E11_{jkq} + \Sigma_{ik} \; B_{ik}\, E12_{ijkq} + \Sigma_i \; C_{ij}\, E13_{ijq} = 0 \qquad 4.7\ a$$

$$\Sigma_{jk} \; A_{jk}\, E21_{ijkq} + \Sigma_k \; B_{ik}\, E22_{ikq} + \Sigma_j \; C_{ij}\, E23_{ijq} = 0 \qquad 4.7\ b$$

$$\Sigma_k \; A_{jk}\, E31_{ijk} + \Sigma_k \; B_{ik}\, E32_{ijk} + C_{ij}\, E33_{ij} = E30_{ij} \qquad 4.7\ c$$

where EIJ are constants resulting from integration, and i,j,k,q range from 1 to N (the value at which the system is truncated). It is easy to eliminate C_{ij} from this system. To eliminate A_{jk} we have to invert N matrices of size N*N. This gives a system in the B_{ik} whose matrix is $N^2 * N^2$. Once solved, we can substitute in 4.6 to get P and hence v. Finally, V can be obtained from 3.1,4. The results are shown in figures 4.1 - 4.5 for Reynolds number = 100 and a/b = 2 ,taking N=5.

5 - CONCLUSION

In this work, we have considered the regular reductive method, which is an integral equation method proposed earlier (SABRY 1984) to solve creeping flows governed by the Stokes' equation with or without heat transfer.

A new transformation is proposed in order to extend the domain of applications of this method to flows having a non-negligible (though a non-dominant) convective term. This is achieved by linearizing the convective term and applying a new transformation of the resulting linearized Navier-Stokes equation into the vector Helmholtz equation, which can be solved using the regular reductive method.

an example, this new technique has been applied to the study of the flow field in the entrance zone of a rectangular duct of infinite length. The results conform with well established empirical results of this classical problem.

As a future research proposal, it is suggested to study thoroughly ·e convergence rate and the effect of the neglected part of the non- ·near convective term

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APPENDIX

MORSE & FESHBACH (1953) have presented a method to find the Green's function of the vector Helmholtz equation in special geometries, where the boundaries are formed by coordinate surfaces. One of these coordinates (x_1) will be called privileged. The coordinates for which this method is applicable are :

* Cartesian coordinates, x_1 could be x, y or z.
* Circular, elliptic or parabolic cylindrical coordinates, x_1 being the axial distance z
* Spherical or conical coordinates, x_1 being the distance from the origin r.

The governing equations are :

$$(\nabla^2 + K^2)\, G(r , r') = I\, \delta\, (r - r') \quad \text{in } \Omega \qquad \text{A.1 a}$$

$$C\,[\, G\,] = 0 \quad \text{on } \delta\Omega \qquad \text{A.1 b}$$

where I is the unitary tensor. The solution takes the form :

$$G(r,r') = -\Sigma_{q=1}^{3}\Sigma_{n=1}^{\infty} F_{qn}(r) \otimes F_{qn}(r') \,/\, (A_{qn}(K_{qn}^2 - K^2)) \qquad \text{A.2 a}$$

where

$$F_{1n} = \nabla\, \Phi_{1n} \qquad \text{A.2 b}$$

$$F_{2n} = \nabla * (a_1\, \mu(x_1)\, \Phi_{2n}) \qquad \text{A.2 c}$$

$$F_{3n} = \nabla * [\nabla * (a_1\, \mu(x_1)\, \Phi_{3n})] \qquad \text{A.2 d}$$

$$A_{qn} = \int_{r\in\Omega} F_{qn} \cdot F_{qn}\, dv \qquad \text{A.2 e}$$

$\mu(x_1) = 1$ (for cartesian and cylindrical coordinates)
x_1 (for spherical and conical coordinates) A.2 f

$\otimes$ represents the tensor product.

Index n represents a trio of number i,j and k, while Φ_{qn} and K_{qn} are eigenfunctions and eigenvalues of:

$$(\nabla^2 + K_{qn}^2)\, \Phi_{qn} = 0 \qquad \text{A.3}$$

The boundary conditions on Φ_{qn} are such as to make F_{qn} satisfy A.1 b. Applying the above method to our problem gives, after summing over index k, the sought for green's function:

$$\bar{G} = \Sigma_{ij} \{ \, \mathbf{i}\mathbf{i} \, X_i'(x) \, Y_j(y) \, X_i'(x') \, Y_j(y') \, G_{ij}(z,z') + \mathbf{j}\mathbf{j} \, X_i(x) \, Y_j'(y) \, X_i(x') \, Y_j'(y') \, G_{ij}(z,z') + \mathbf{j}\mathbf{j} \, X_i(x) \, Y_j(y) \, X_i(x') \, Y_j(y') \, G_{ij}'(z,z') \, \} \qquad \text{A.4 a}$$

where $X_i'(x) = \sqrt{(2/a)} \cos((2i-1)\pi x/a)$ A.4 b

$Y_j'(y) = \sqrt{(2/b)} \cos((2j-1)\pi y/b)$ A.4 c

$G_{ij}(z,z') = -\sinh(\beta_{ij} z) \exp(-\beta_{ij} z')/\beta_{ij}$ for $z'>z$

or $-\exp(-\beta_{ij} z) \sinh(\beta_{ij} z')/\beta_{ij}$ for $z>z'$ A.4 d

$G_{ij}'(z,z') = -\cosh(\beta_{ij} z) \exp(\beta_{ij} z')/\beta_{ij}$ for $z'>z$

or $-\exp(-\beta_{ij} z) \cosh(\beta_{ij} z')/\beta_{ij}$ for $z>z'$ A.4 e

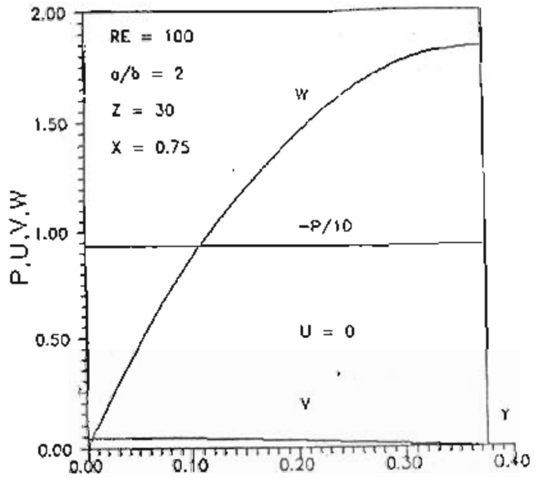

FIGURE 4.1

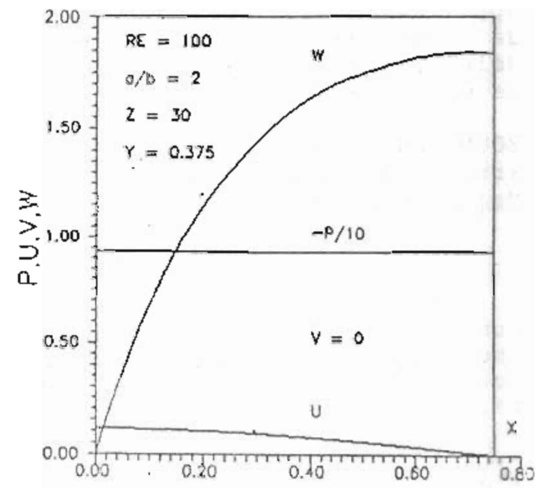

FIGURE 4.2

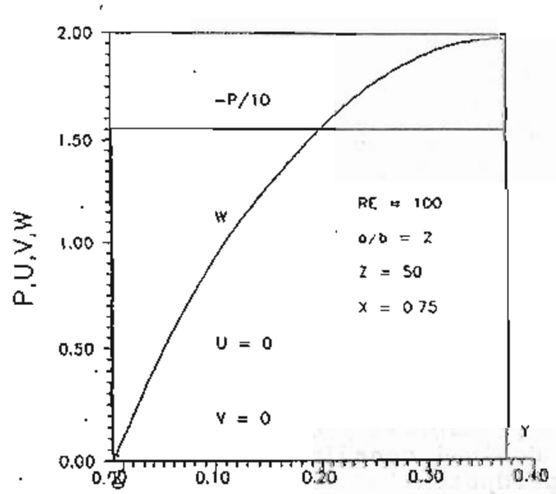

FIGURE 4.3