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Generalization of Finite Integral Transforms for Treating Nonlinear Problems in Heat Diffusion. Part II: Application to a Nonlinear Case.

Bishri Abdel-Hamed
Assistant Professor of Applied Mathematics and Physical Sciences Engineering Department, Mansoura University, Egypt.

GENERALIZATION OF FINITE INTEGRAL TRANSFORMS FOR TREATING NONLINEAR PROBLEMS IN HEAT DIFFUSION.

PART II: Application to a nonlinear case

تعميم التحويل التكاملي المحدود لحل
معادلات التوصيل الحراري الغير خطية
الجزء الثاني – تطبيق

Bishri Abdel-Hamid
Assistant Professor
Applied Mathematics and Physical Sciences Dept.,
Al-Mansoura University, Egypt

يقدم البحث طريقة التحويل التكاملي الموحدة ، والتي اقترحها الباحث وقدمها في بحوث سابقة ، مع تطبيقها على مسائل السريان الحراري في حالة التعرض لتبادل حراري مركب بالحمل والاشعاع وهي حالة يصعب فيها استخدام طرق الحل التقليدية نظرا لان المعادلة التفاضلية الجزئية التي تصف هذه الظاهرة خاضعة لشروط حدية غير خطية مما ادى الى تعذر ايجاد حلول لها على الطرق التقريبية .

والخطوات المتبعة للحصول على الحل باستخدام طريقة التحويل التكاملي الموحدة في الحالات الغير خطية مشابهة للخطوات المتبعة في حل المعادلات الخطية مع معالجة خاصة للحدود الغير خطية ، وقد امكن باستخدام الطريقة المقترحة الحصول على حل تحليلي والذي اكد جدوى طريقة التحويل التكاملي الموحدة عند مقارنة لنتائج طريقة الفروق المحددة العددية .

ABSTRACT

A solution methodology, based on the finite integral transform technique, appropriate for solving nonlinear problems of heat diffusion was developed by the author in previous work [1,2]. In this paper, the method is applied to solve the heat diffusion problem in a finite region subject to nonlinear boundary conditions due to radiation exchange at the interface according to the fourth power law. The results obtained form this analytic solution are compared with those obtained from a numerical solution developed using an explicit finite difference method.

INTRODUCTION

Heat diffusion problems with nonlinear boundary condltions arise in many practical situations [3-7]. In particular, the nonlinear boundary condition appears in combustion systems [3], where in the pre-ignition heating, the particle entering a furnace and travelling toward a flame front receives heat uniformly by thermal radiation from the furnace walls and loses heat uniformly by convection to the surrounding gases. It appears also in flash heating of powdered solids in mineral processing industries [4], where particles are heated by convection and as their temperature rises they begin to lose heat by thermal radiation. In nuclear technology, heat transfer is dominated by boiling, thermal radiation and forced convection [5]; therefore, the heat transfer coefficients depend on the surface temperature and thus the boundary conditions become nonlinear. Similar boundary conditions appear in attic radiant barrier and other applications [6-8].

The analytic solutions obtained from nonlinear heat diffusion equations differ significantly from solution obtained from the, by assumption, linearized equations. Unfortunately, few analytic solutions for nonlinear cases of heat diffusion have appeared. The inherent nonlinearity of these problems has limited analytical investigations to extremely simplified cases [9,10]

The finite integral transform technique has been applied to linear homogeneous and nonhomogeneous problems [9,11,12] and then applied to nonlinear heat conduction problems with variable thermal conductivity [13,14]. However, the method has not been yet directly applied to problems with nonlinear boundary conditions.

Recently, a methodology based on the finite integral transform for solving nonlinear diffusion problems has been developed by the author [1]. The methodology is demonstrated on a heat diffusion problem with nonhomogeneous separable boundary conditions and the solution was compared to the exact solution in [2]. The same methodology is extended in this paper to solve the problem of heat diffusion in a finite region subject to nonlinear boundary conditions resulting from a coupled convection and radiation exchange at the surface according to the "fourth power law".

PROBLEM DESCRIPTION

A one-dimensional finite slab of thickness L is considered. The slab has one surface insulated and the other surface transfers heat to a convecting medium maintained at T_∞ and having a heat transfer coefficient h. At the same time, there is heat exchange by radiation between the surface and the enclosure which is maintained at T_r. The convection coefficient, h, the surface emissivity, ϵ and the thermophysical properties of the solid are assumed invariant. Finally, the initial temperature is assumed to be uniform throughout the solid.

The above transient heat conduction problem can be described by the following partial differential equation

$$\frac{\partial^2 T}{\partial x^2}(x,t) = \frac{1}{\alpha}\frac{\partial T}{\partial t}(x,t); \qquad 0<x<L, \quad t>0, \tag{1}$$

subject to the following boundary conditions;

$$k\frac{\partial T}{\partial x}(x,t) - h(T(x,t)-T_\infty) = \sigma\epsilon(T^4(x,t)-T_r^4); \qquad x=0, \quad t>0 \tag{2a}$$

$$\frac{\partial T}{\partial x}(x,t) = 0; \qquad x=L, \quad t>0. \tag{2b}$$

and the initial condition;

$$T(x,t) = T_o. \qquad t=0, \quad 0<x<L. \tag{2c}$$

For convenience, we recast the above governing equation and the auxiliary conditions into the following dimensionless form

$$\frac{\partial^2\theta}{\partial\eta^2}(\eta,\tau) = \frac{\partial\theta}{\partial\tau}(\eta,\tau) , \tag{3}$$

subject to the following dimensionless boundary conditions;

$$\frac{\partial\theta}{\partial\eta}(\eta,\tau) - \beta\theta(\eta,\tau) = \Pi(\eta,\tau,\theta) ; \qquad \eta=0, \quad \tau>0 \tag{4a}$$

$$\frac{\partial\theta}{\partial\eta}(\eta,\tau) = 0 ; \qquad \eta=1, \quad \tau>0 \tag{4b}$$

and the dimensionless initial condition

$$\theta(\eta,\tau) = 1. \qquad \tau=0, \quad 0<\eta<1 \tag{4c}$$

The dimensionless parameters η, τ, β and θ are defined in the nomenclature. The right-hand side of equation (4a) is

$$\Pi(\eta,\tau,\theta) = \Omega\left[\theta^4(\eta,\tau) - \theta_r^4\right] - \beta\theta_\infty \tag{6}$$

where Ω is the radiation constant and is defined as

$$\Omega = \frac{\sigma\epsilon T_o^3 L}{k} \tag{7}$$

where σ, ϵ and k are the Stefan-Boltzman constant, the surface emissivity, and the thermal conductivity, respectively.

FINITE INTEGRAL TRANSFORM SOLUTION

The finite integral transform method is applied to determine the temperature distribution $\theta(\eta,\tau)$. The method proceeds by treating the nonlinearity as an effective nonhomogeneouity. The associated eigenvalue problem is developed from considering the associated linear homogeneous part of the dimensionless problem given by (3) and (4).

Employing the method of separation of variables on the associated linear homogeneous problem yields the following eigenvalue problem

$$\frac{\partial^2 \psi}{\partial \eta^2} + \lambda_n^2 \psi(\eta) = 0, \tag{8a}$$

subject to

$$\frac{\partial \psi}{\partial \eta} - \beta \psi(\eta) = 0; \qquad \eta = 0 \tag{8b}$$

$$\frac{\partial \psi}{\partial \eta} = 0 \; ; \qquad \eta = 1. \tag{8c}$$

The solution to this problem gives the eigenfunctions

$$\psi_n(\eta) = \lambda_n \cos\lambda_n\eta + \beta \sin\lambda_n\eta, \qquad n=1,2,... \tag{9}$$

which correspond to the discrete set of eignvalues given by the transcendental equation

$$\beta \cos\lambda_n - \lambda_n \sin\lambda_n = 0, \quad n=1,2,... \tag{10}$$

The orthogonality relation for the eigenfunction are

$$\int_0^1 \psi_n(\eta)\psi_m(\eta)d\eta = \begin{cases} 0, & m \neq n \\ N(\lambda_n), & m = n \end{cases} \tag{11}$$

where $N(\lambda_n)$ is the normalization integral given by

$$N(\lambda_n) = \frac{1}{2}\{ (\lambda_n^2+\beta+\beta^2) + \frac{1}{2\lambda_n}(\lambda_n^2-\beta^2)\sin 2\lambda_n - \beta\cos 2\lambda_n \}. \qquad n=1,2,... \tag{12}$$

The integral transform pair is developed using the orthogonality relation and may be readily written as [9];

INTEGRAL TRANSFORM

$$\Phi(\lambda_n,\tau) = \int_0^1 \psi_n(\eta)\theta(\eta,\tau)\, d\eta. \tag{13a}$$

INVERSION FORMULA

$$\theta(\eta,\tau) = \sum_{n=1}^{\infty} \frac{\psi_n(\eta).\Phi(\lambda_n,\tau)}{N(\lambda_n)} \tag{13b}$$

Operating on (3) with $\int_0^1 \psi_n(\eta)d\eta$ and on equation (8a) with $\int_0^1 \theta(\eta,\tau)d\eta$, subtracting the results followed by utilizing the definition of the integral transform, yields

$$\int_0^1 \psi_n \frac{\partial^2\theta}{\partial\eta^2}(\eta,\tau)\, d\eta - \int_0^1 \theta(\eta,\tau)\frac{\partial^2\psi_n}{\partial\eta^2}d\eta = \frac{d\Phi}{d\tau}(\lambda_n,\tau)+\lambda_n^2\Phi(\lambda_n,\tau). \quad (14)$$

By tbe use of the second Green's theorem [9] along with the boundary conditions given by equations (4) and equations (8), one can reduce equation (14) to the following system of first order nonhomogeneous ordinary differential equations

$$\frac{d\Phi}{d\tau}(\lambda_n,\tau) + \lambda_n^2\Phi(\lambda_n,\tau) = -\tilde{\Pi}_n(0,\tau,\Phi)\psi_n(0), \quad n=1,2,... \quad (15)$$

subject to the following transformed initial condition

$$\Phi(\lambda_n,0) = \int_0^1 \psi_n(\eta)\theta(\eta,0)d\eta, \qquad n=1,2,... \quad (16)$$

where,

$$\tilde{\Pi}_n(\eta,\tau,\Phi) = \Pi\left(\eta,\tau,\sum_{j=1}^{\infty}\frac{\psi_j(\eta)\Phi(\lambda_j,\tau)}{N(\lambda_j)}\right) \quad (17)$$

and where the inversion formula given by equation (13b) has been utilized.

The solution of equation (15) renders the dimensionless integral transform, $\Phi(\lambda_n,\tau)$. The solution can be obtained using an appropriate numerical integration scheme. Once $\Phi(\lambda_n,\tau)$ is obtained, the temperature diatribution can be reconstructed through the use of the inversion formula.

RESULTS

The transient, nonlinear heat conduction problem subject to a radiation boundary condition is investigated for two cases. A linear case characterized by no heat exchange by radiation at the boundary, i.e. $\Omega = 0$, followed by a case of stronger nonlinearity when $\Omega=2$. The results of the generalized finite integral transform are compared to those obtained from a finite difference scheme.

In the first case, all the heat exchange at front surface is due to convection. The temperature decay increases as Biot number grows as shown in Figures (1) and (2). While the finite difference method predicts higher values, the maximum difference in dimensionless temperature predicted form the two solution methods is within 0.6%.

In the second case, the effect of a stronger nonlinearity is considered by taking $\Omega=2$ and doubling the temperature of the enclosure θ_r. As shown in Figures (3) and (4) the solid's temperature response under these circumstances is dominated by the radiation exchange and the effect of Biot number is not as obvious as in the first case. The finite difference method appears to predict lower values of temperature at the back boundary. A maximum difference of 1.3% between the two solutions was noticed in this case as shown in Table (I).

In both cases the back boundary responds slower to thermal effects on the front boundary. However, the temperature of both surfaces approach each other as time progresses until steady-state is reached.

CONCLUSIONS

The solution of a transient heat conduction problem subject to nonlinear boundary conditions due to a coupled convection-radiation heat exchange has been obtained. It was shown that the generalized finite integral transform provides a straightforward methodology for heat equations subject to nonlinear boundary conditions. Combining this result with the results reported in literature for other types of nonlinearities, it is concluded that the finite integral transform technique is a general methodology that can be used to solve both linear and nonlinear heat conduction problems. The method is systematic, consistent, and computationally rapid.

NOMENCLATURE

b	Film thickness
c	Specific heat
f	A subscript denotes fluid
h	Heat transfer coefficient between the film and the surrounding
k	Thermal conductivity of the slab
L	Thickness of a slab
$N(\lambda_n)$	Normalization integral
t	Dimensional time variable
T_∞	Ambient temperature
$T(x,t)$	Dimensional Temperature
$W(\eta)$	A weight function
x	Space variable
s	A subscript refers to solids

Greak Symbol

α	Thermal diffusivity
$\beta = \frac{hL}{k}$	Biot number
$\Phi(\lambda_n, \tau)$	Transform of dimension less temperature
$\eta = \frac{x}{L}$	Dimensionless space variable
λ_n	Eigenvalues
θ_∞	Dimensionless temperature of the surrounding
$\theta(\eta,\tau)=\frac{T(x,t)}{T_o}$	Dimensionless temperature
ρ	Density of solid
$\tau = \frac{\alpha t}{L^2}$	Dimensionless time
$\psi_n(\eta)$	Eigenfunctions
Γ	Heat capacity ratio

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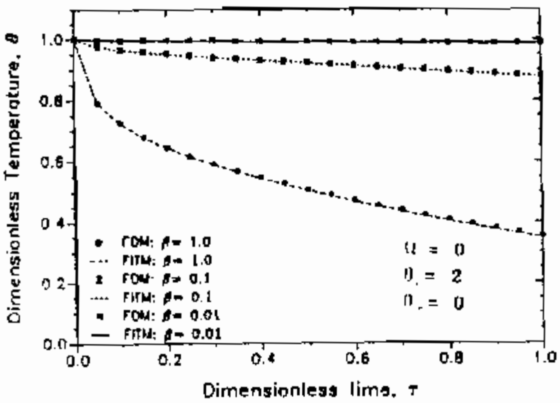

Fig.(1) Dimensionless Temperature–Time Variation at Front Surface for Different Biot Numbers

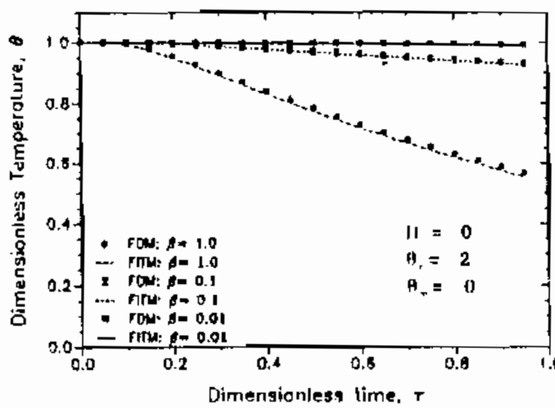

Fig (2) Dimensionless Temperature–Time Variation at Back Surface for Different Biot Numbers

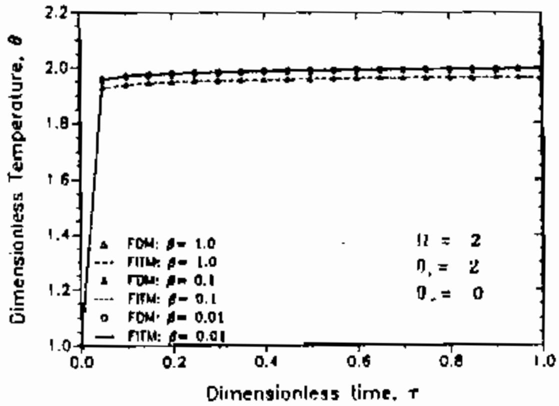

Fig.(3) Dimensionless Temperature–Time Variation at Front Surface for Different Biot Numbers

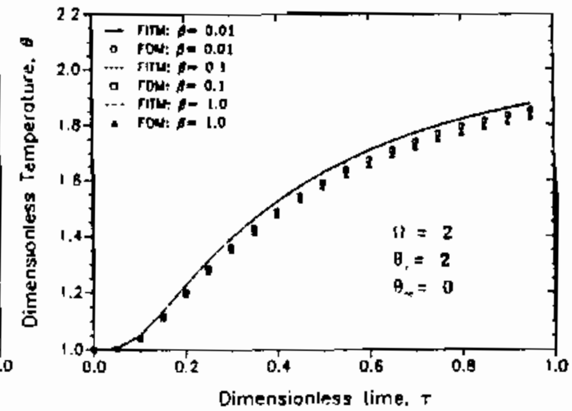

Fig.(4) Dimensionless Temperature–Time Variation at Back Surface for Different Biot Numbers

TABLE (1)

RELATIVE DIFFERENCE IN TEMPERATURE CALCULATED FROM THE FINITE DIFFERENCE AND THE FINITE INTEGRAL TRANSFORM METHODS

CASE - 1

% Relative Difference

TIME	TEMP. (FITT)	FRONT SURFACE FD	TEMP. (FITT)	BACK SURFACE FD
Biot = 0.01, Rad. Const. = 0, Trad = 2.0, and Tamb = 0				
0.000	1.0000000	0.0000000	1.0000000	0.0000000
0.100	0.9964700	0.0050126	0.9999500	0.0039997
0.200	0.9950300	0.0110520	0.9994400	0.0070075
0.300	0.9938900	0.0150887	0.9986500	0.0110179
0.400	0.9928600	0.0201472	0.9977500	0.0150303
0.500	0.9918700	0.0241935	0.9968000	0.0180584
Biot = 0.1, Rad. Const. = 0, Trad = 2.0, and Tamb = 0				
0.000	1.0000000	0.0000000	1.0000000	0.0000000
0.100	0.9652900	0.0010374	0.9992200	-0.0040026
0.200	0.9514200	0.0031512	0.9940000	-0.0221329
0.300	0.9406200	0.0021291	0.9860700	-0.0395503
0.400	0.9309600	-0.0032205	0.9771700	-0.0542387
0.500	0.9217800	-0.0108504	0.9679800	-0.0661145
Biot = 1, Rad. Const. = 0, Trad = 2.0, and Tamb = 0				
0.000	1.0000000	0.0000000	1.0000000	0.0000000
0.100	0.7235800	0.0027678	0.9931100	-0.0493109
0.200	0.6433900	-0.0031128	0.9506400	-0.2219562
0.300	0.5888500	-0.0424526	0.8918000	-0.3834935
0.400	0.5441700	-0.1176057	0.8309500	-0.5222929
0.500	0.5045200	-0.2160450	0.7725300	-0.6472234

CASE -2

% Relative Difference

TIME	TEMP. (FITT)	FRONT SURFACE FD	TEMP. (FITT)	BACK SURFACE FD
Biot = 0.01, Rad. Const.= 2.0, Trad= 2.0, and Tamb= 0				
0.000	1.0000000	0.0000000	1.0000000	0.0000000
0.100	1.9711601	-0.0020260	1.0471100	0.4135115
0.200	1.9799000	0.0010115	1.2189900	0.9401182
0.300	1.9844700	0.0055446	1.3826100	1.0523554
0.400	1.9877900	0.0090556	1.5144800	1.0260950
0.500	1.9903600	0.0105532	1.6184700	0.9527493
Biot = 0.1, Rad. Const.= 2.0, Trad = 2.0, and Tamb = 0				
0.000	1.0000000	0.0000000	1.0000000	0.0000000
0.100	1.9683000	-0.0045728	1.0470099	0.4164202
0.200	1.9770600	0.0000000	1.2185200	0.9519759
0.300	1.9816300	0.0045419	1.3818001	1.0725212
0.400	1.9849600	0.0080596	1.5134100	1.0512702
0.500	1.9875300	0.0100644	1.6171900	0.9819494
Biot = 1.0, Rad. Const.= 2.0, Trad= 2.0, and Tamb = 0				
0.000	1.0000000	0.0000000	1.0000000	0.0000000
0.100	1.9394500	-0.0304193	1.0459800	0.4445589
0.200	1.9483100	-0.0205279	1.2137400	1.0710706
0.300	1.9529300	-0.0128004	1.3735800	1.2674851
0.400	1.9583000	-0.0071539	1.5024700	1.2991970
0.500	1.9589000	-0.0035722	1.6041400	1.2673502