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Optimal Control Implementation for Linear Systems Via Riccati Equation

تحقيق وتطبيق نظام تحكم امثل للانظمة الخطية باستخدام معادلات ريكاتي

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يقدم البحث تحقيق وتصميم نظام تحكم امثل للانظمة الديناميكية الممثلة رياضيا في صورة خطية ، وذلك عن طريق دالة التكلفة - أو دالة الطاقة cost function - في صيغة تربيعية للاستخدام في تنفيذ طريقة "بونترياجون العظمى " ومن ثم الاستعانة بمعادلات "ريكاتي " لاستكمال اجراء التصميم الامثل .

واذ يهتم البحث بصفة خاصة بحالات التنظيم -او التنفيذ regulating problems التي تستهدف بالدرجة الاولى الحفاظ على المستوى الصفري المطلوب لمتغيرات الحالة ، فانه يتعرض لتطبيق هذه الطرق على كل من الانظمة المستمرة (continuous) التي توصف فيها ديناميكية النظام بمجموعة معادلات خطية تفاضلية ، والانظمة المتقطعة - او المنفصلة discrete - التي توصف فيها ديناميكية النظام بمعادلات الفروق (difference equations)

وقد تم اختبار الطرق المستعرضة وذلك بتطبيقها على مثال عددي من الدرجة الرابعة يمثل حالة عملية لتلوث فرعي مجرى مائي، وتمت مقارنة النتائج المحسوبة للمسألة في كل من حالتيها المتصلة والمنفصلة للتأكيد على مميزات الصيغة المتقطعة للتمثيل الرياضي ، والتي تتمتع بالطبع بمميزات كبيرة تزداد استخدامات الحاسبات الرقمية (digital computers)

Abstract:

The paper deals with the implementation of optimal controller design for linear dynamic systems. The implementation is carried out with the use of quadratic cost function for both cases of continuous and discrete systems. The mathematical optimal control theory is applied to dynamical systems with considering the form of either vector differential equations in continuous time case, or difference equations in discrete time- or sampled data-case.

The bulk of the work provides design techniques for synthesizing the optimal control structure by applying the Pontryagin's Maximum principle to linear systems with using Matrix Riccati Equation. The work is focused upon regulator problems; that is problems where the goal is to maintain the system states at zero level.

The implemented techniques are tested by the application on fourth order practical example represents the linearized control model of two-reach river pollution system. Also comparative study between the continuous and discrete optimal regulator problems will be presented and discussed from a numerical point-of-view.

Introduction:

During the past twenty years, significant amount of researches have been carried out in the area of system optimization. In this field, Optimal Control Theory is a new and direct approach which has the ability to handle analysis and synthesis of complicated control problems. This approach has been made feasible with the development and advance of digital computers.

The mathematical theory of optimal control is applied to dynamical systems; which take the form of either vector differential equations (in continuous-time case) or deference equations (in discrete-time or sampled-data case). The two main theoretical approaches to the optimization control are:

1-Bellman's dynamic programming method which is based on the principle of optimality [2]. It is referred to as the Hamilton-Jaccobi-Bellman Theory. Its major disadvantage is the large computer memory requirements.

2-Pontryagin's maximum principle which is an extension and application of the classical calculus of variations to the optimal control [5]. Its major disadvantage is that it provides, in general, only local necessary conditions for optimality. But from other side, its computational requirements are not complicated as that in case of dynamic programming.

The work implemented here is restricted to the application of the Pontryagin's maximum principle to linear control systems via the use of Matrix Riccati Equation. Also this work is focused upon regulator problems, that is problems where the goal is to maintain the states of the system at zero level. The main two restrictions considered for this study are:

1-The complete states of the plant can be accurately measured at all times and are available for feedback.

2-The cost function is chosen to be in quadratic form in order to simplify the computations. Moreover, quadratic construction is enough for linear control systems design.

A fourth order numerical example is considered for applying the implemented techniques, and for concluding the numerical comparison between the continuous and discrete time cases under study.

Continuous Linear Optimal Control Technique:

For the plant described by the following linear state equation :

$$\dot{X}(t) = A\,X(t) + B\,U(t)\ , \qquad X(t_o) = x_o \tag{1}$$

where : X(t) is (nx1) state vector
U(t) is (mx1) control vector
A is (nxn) system coefficients matrix
B is (nxm) control matrix

the cost function to be minimized is: [6,7]

$$J = \frac{1}{2} X^T(t_f) W X(t_f) + \frac{1}{2}\int_{t_o}^{t_f} \left[X^T(t) Q X(t) + U^T(t) R U(t) \right] dt \quad (2)$$

where : Q , W are (nxn) real symmetric positive semi-definite matrices.
R is (mxm) real symmetric positive definite matrix.
t_f is the terminal time (fixed value).

Solving this optimality problem via Pontryagin's Maximum Principles[6], the Hamiltonian equation is:

$$H [X(t), U(t), P(t), t] = \frac{1}{2}\left[X^T(t) Q X(t) + U^T(t) R U(t) \right] + P^T(t)\left[A X(t) + B U(t) \right] \quad (3)$$

where : P(t) is (nx1) Lagrange Multiplier Vector

Application of maximum conditions produces:

$$\partial H/\partial U(t) = 0 = R U(t) + B^T P(t) \quad (4)$$

and

$$\partial H/\partial X(t) = -\dot{P}(t) = Q X(t) + A^T P(t) \quad (5)$$

with terminal condition:

$$\partial H/\partial X(t_f) = P(t_f) = W X(t_f) \quad (6)$$

For performing closed loop control, the solution of P(t) is considered as a linear function of the states , and is constructed similar to the terminal condition given by equation (6) as a function of the (nxn) Riccati symmetric matrix K(t) as follows :

$$P(t) = K(t) X(t) \quad (7)$$

for which ; the weighted terminal condition matrix will be:

$$K(t_f) = W \quad (8)$$

Thus from equations (4) and (7), the optimal linear control law will be :

$$U(t) = G(t) X(t) \quad , \quad G(t) = -R^{-1}B^T K(t) \quad (9)$$

where : G(t) is (mxn) time varying gain matrix .

The final linear state model of the controlled plant will be obtained by substituting equation (9) into equation (1) giving :

$$\dot{X}(t) = A X(t) + B G(t) X(t) \quad , \quad X(t_o) = x_o \quad (10)$$

Now, for calculating the Riccati Matrix K(t), Differential Matrix Riccati Equation (DMRE) can be constructed by solving equations (5) , (7) and (10) giving :

$$\dot{K}(t) = - K(t)\ A - A^T K(t) - K(t)\ B\ G(t) - Q \qquad (11)$$

It should be noted that , the Riccati Matrix K(t) is independent of the initial conditions. So; it can be pre- calculated separately by backward integration in time starting from the defined terminal time (t_f) and terminal condition given by equation (8).

Since U(t) is a linear function of the current states X(t) , and G(t) is independent of the initial conditions , so the states will converge to zero whatever the value of the initial states are, and a closed loop controller is obtained as represented schematically in figure (1). Figure (2) shows the flow chart representing the corresponding calculation procedure .

As an important restriction , it is notable that the complete states X(t) of the plant have to be accurately measured at all time, and have to be available for feed - back.

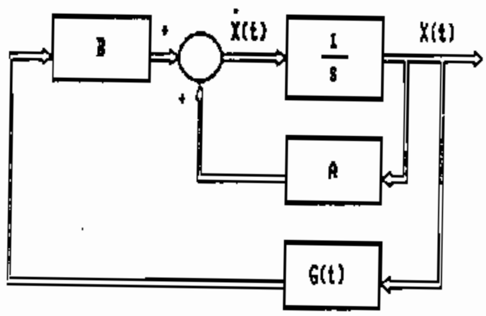

Fig. (1) Continuous optimal linear regulator problem.

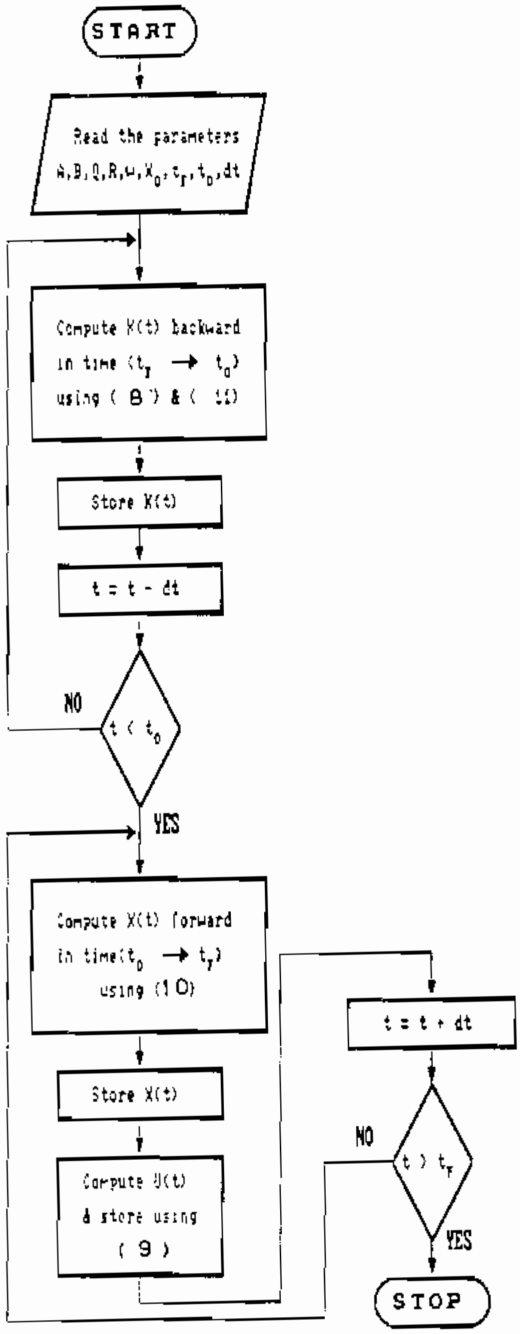

Fig. (2) Flow-chart for continuous optimal linear regulator problem.

Discrete Linear Optimal Control Technique:

For discrete linear system , closely parallel steps to that of the continuous systems technique can be carried out with improving the necessary mathematical modifications. So; brief steps are summarized as follows :
For plant of the following discrete state space form:

$$X(k+1) = C\,X(k) + D\,U(k) \quad , X(o) = x_o \quad , k = 0,1,2,\ldots\ldots,N \qquad (12)$$

where C and D can be calculated for sampling period (T) as the following forms [4]:

$$C = e^{AT} \qquad (13)$$

$$D = \left(\int_o^T e^{AT}\,dt \right) B \qquad (14)$$

the cost function is :

$$J = 1/2\, X^T(k)\, W\, X(k) + 1/2 \sum_{k=0}^{N-1} \left[X^T(k)\, Q\, X(k) + U^T(K)\, R\, U(k) \right] \qquad (15)$$

and the Hamiltonian takes the form :

$$H\left[X(k)\,,\, U(k),\, P(k),\, k \right] = 1/2 \left[X^T(k)\, Q\, X(k) + U^T(k)\, R\, U(k) \right] + P^T(k+1) \left[C\,X(k) + D\,U(k) \right] \qquad (16)$$

Application of maximum conditions produces:

$$O = R\,U(k) + D^T P(k+1) \,, \qquad (17)$$

$$P(k) = Q\,X(k) + C^T\,P(k+1) \qquad (18)$$

with suggesting the Riccati discrete solution :

$$P(k) = F(k)\,X(k) \qquad (19)$$

where the terminal boundary condition is :

$$F(N) = W \qquad (20)$$

By solving equations (17) to (19) ;the optimal linear closed loop control law will be :

$$U(k) = G(k)\,X(k) \qquad , G(k) = -R^{-1}\, D^T\, (C^{-1})^T \left[F(k) - Q \right] \qquad (21)$$

where : G(k) is the (mxn) gain matrix.
Back substitution of equation (21) into equation (12) produces the following controlled linear discrete state model :

$$X(k+1) = C\,X(k) + D\,G(k)\,X(k) \quad , X(o) = x_o \quad ,k = 0,1,2,\ldots\ldots N \qquad (22)$$

From equations (12), (17) , (18) and (19); the Difference Matrix Riccati Equation will be :

$$F(k) = Q + C^{T} F(k+1) \left[I + D R^{-1} D^{T} F(k+1) \right]^{-1} C \qquad (23)$$

where : I is the identify matrix.

Similar to the continuous technique , the Riccati matrix F(k) given by equation (23) can be pre - calculated separately by backward substitution starting from the terminal condition given by equation (20).

With very little symbolic modification, similar block diagram and flow chart for that shown in figures (1) and (2) can be simply constructed for representing this discrete technique.

Application Example and Results:

Considering the application example representing the two-reach model of a river pollution control system [1] , which is represented by the following fourth order linearized state model:

$$\dot{X}(t) = \begin{bmatrix} -1.32 & 0.0 & 0.0 & 0.0 \\ -0.32 & -1.2 & 0.0 & 0.0 \\ 0.9 & 0.0 & -1.32 & 0.0 \\ 0.0 & 0.9 & -0.32 & -1.2 \end{bmatrix} X(t) + \begin{bmatrix} 0.1 & 0.0 \\ 0.0 & 0.0 \\ 0.0 & 0.1 \\ 0.0 & 0.0 \end{bmatrix} U(t) \qquad (24)$$

for designing a continuous linear regulating problem, it is desired to find optimal controls which minimize the quadratic cost function described by equation(2) with the following parameters:

Q = Diag.(2,4,2,4) , R = Diag.(2,2) ,
W = 0.0 , t_o = 0.0 , t_f = 10 sec.

A fourth order Differential Matrix Riccati Equation (DMRE) is solved by integrating equation (11) backward in-time using a fourth order Runge-Kutta method [3] for a step size dt=0.1 sec., with the boundary conditions $K(t_f) = 0.0$. The solution of this equation is shown in figures (3-a) and (3-b) (noting that, these figures are plotted backward in time).

Then, optimal states and optimal controls are solved by using equations (9) and (10) through similar Runge-Kutta Technique for substituted initial conditions X(0) = [1 1 -1 1]. These solutions are illustrated in figures (3-c) and (3-d) respectively.

For discrete linear regulating problem , the fourth order discrete state model can be deduced from equations (13),(14) and (24) for T=0.1

This model will be :

$$X(k+1) = \begin{bmatrix} 0.876341 & 0.0 & 0.0 & 0.0 \\ -0.02821184 & 0.8869204 & 0.0 & 0.0 \\ 0.07887068 & 0.0 & 0.976341 & 0.0 \\ -0.00253906 & 0.07982284 & -0.02821184 & 0.8869204 \end{bmatrix} X(k)$$

$$+ \begin{bmatrix} 9.371201\ E{-03} & 0.0 \\ -1.464693\ E{-04} & 0.0 \\ 4.103210\ E{-04} & 9.371201\ E{-03} \\ -8.673632\ E{-06} & -1.464693\ E{-04} \end{bmatrix} U(k) \qquad (25)$$

The discrete problem is solved for the same weighted matrices of the cost function, the initial conditions, and initial and final times as in the continuous case. Figures (4-a) and (4-b) show the plot of the gain matrix G(k), and figures (4-c) and (4-d) show the optimal statesand controls respectively. All numerical calculations are carried out using OLIVETTI M240 Personal Computer.

Conclusions :

1- The situation wherein the process is to be controlled for an interval of infinite duration merits special attention. That is for completely controllable plant,with zero terminal conditions (W=0.0), and constant system and weighting cost function matrices (A,B,R,and Q are constants), the Riccati Matrix becomes a constant matrix as t_f approaches large time (little than 6 seconds in our example).Then, the optimal control law is stationary, and the controller consists of m-fixed summing amplifiers, each having n-inputs.

2- Responses of continuous and discrete optimal controllers are very close.But of course, discretization process makes the results less accurate (depending on the chosen sampling period)

3- Execution time required to solve the problem is reduced from about 9 minuets in continuous case to about 2.5 minuets in discrete case. So, it is recommended to follow the discrete procedure which does not need any integration method (like Runge-Kutta in continuous systems).

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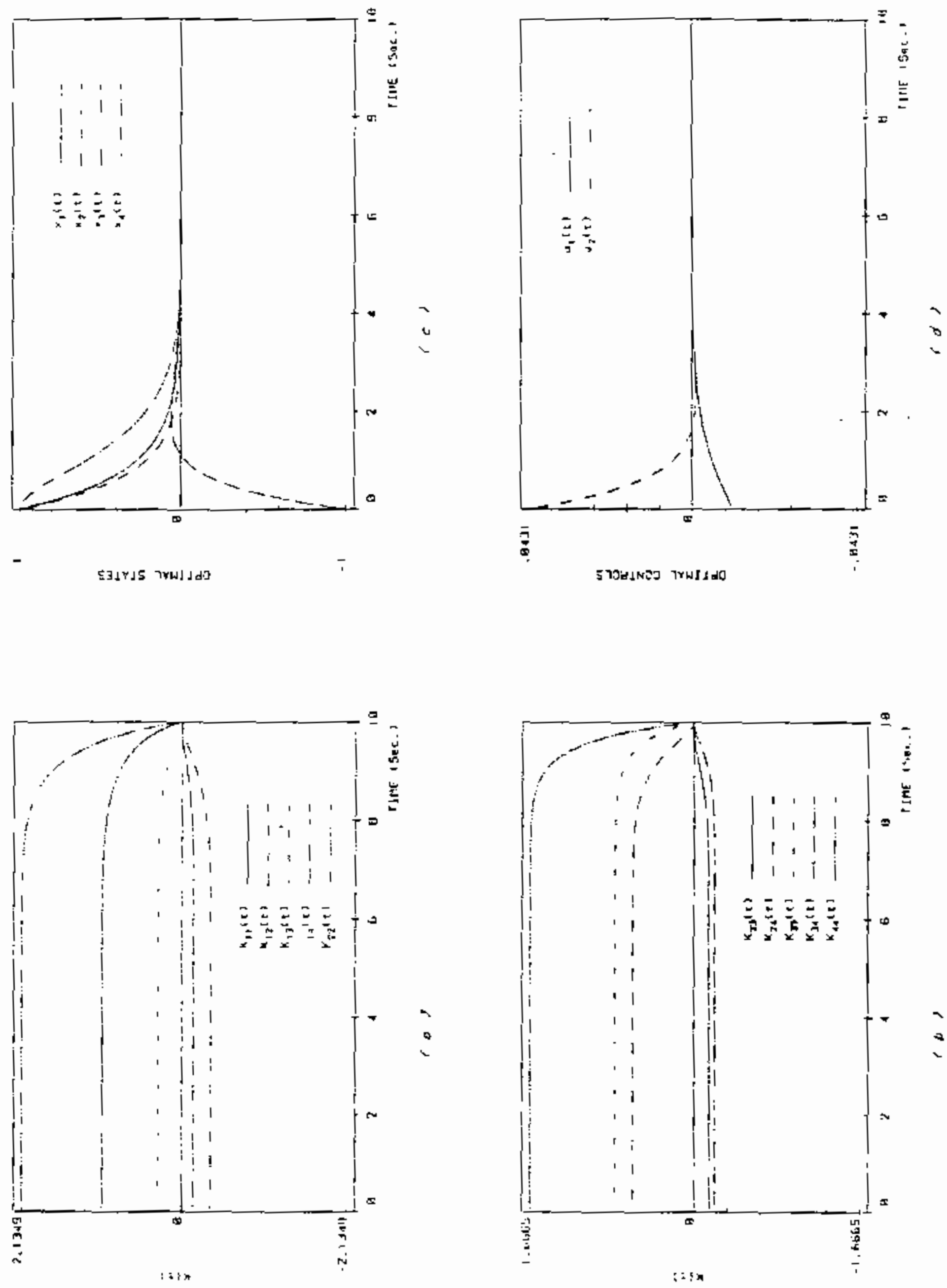

Fig. (3) The optimal solution for the example : (a),(b) the solution of the DMRE, (c) the states, (d) the controls.

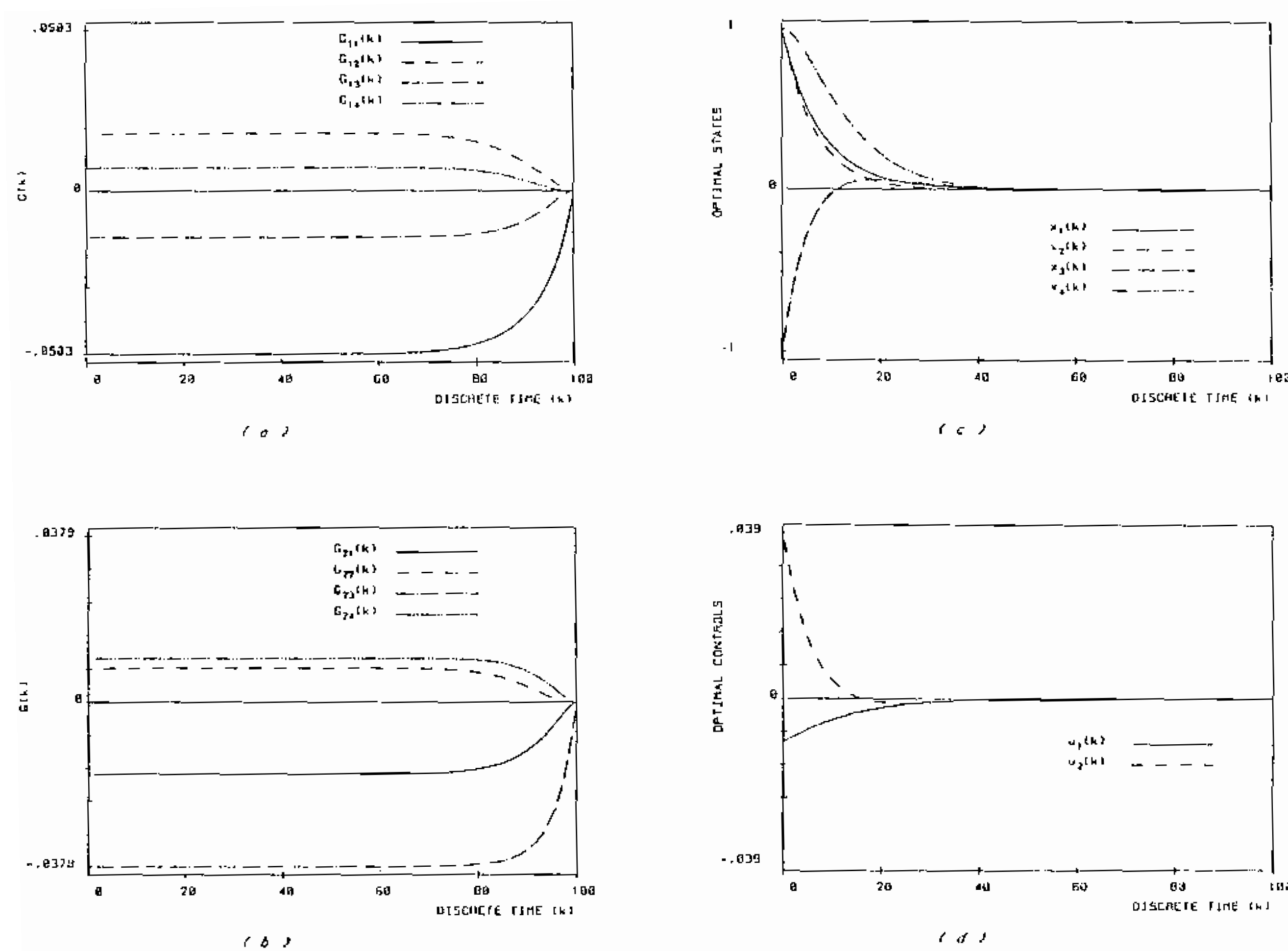

Fig. (4) The DISCRETE optimal solution : (a),(b) the plot of the variable gain matrix G(k), (c) the states, (d) the controls.