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ANALYSIS OF TRANSIENT HEAT CONDUCTION IN A 2-D RECTANGULAR FIN

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التوصيل الحرارى الغير مستقر ثنائى البعد فى زعنفه مستطيله

يهدف البحث الى الحصول على التطور الزمنى لتوزيع درجات الحراره فى اتجاهين متعامدين نتيجة لانتقال الحراره بالتوصيل الغير مستقر فى زعنفه مستطيلة المقطع تتعرض قاعدتها فجاه لدرجه حراره ثابته تختلف عن درجة حرارة الوسط المحيط بها، كما افترض ان الزعنفه تبرد بالحمل وان طرفها الحر معزول . ونظرا لصعوبة الحصول على حل رياضى تحليلى لهذه المشكله فلقد تم تقسيمها الى جزئين ، فى الجزء الاول تم الحصول على التطور الزمنى لدرجات الحراره بطريقة الفروق المحدوده ، وفى الجزء الثانى تم الحصول على حل رياضى تحليلى كامل لهذه المشكله فى حالة الاستقرار . وفى اتجاه آخر فقد تم تصحيح علاقه منشوره فى المرجع رقم 4 يمكن بها تحويل المشكله الثنائية الابعاد الى مشكلة احادية البعد . وتدل النتائج على ان عدد Biot يؤثر فى توزيع درجات الحراره وتطورها الزمنى تأثيرا حاسما، كما وجد ان العلاقه المنشوره فى المرجع رقم 4 والتى تم تصحيحها فى هذا العمل لا تعطى حلا صحيحا الا فى ازمنه مبكره ولقيم صغيره من عدد Biot ولا تطبق الا لزعنفه رفيعه . اما الحل المقدم فى هذا العمل هو حلا عاما يصلح لجميع الازمنه ولاى سمك ولاى عدد Biot .

ABSTRACT

In this work a new developed dimensionless finite difference technique for 2-D transient heat conduction is developed and presented. The technique is then applied to predict the time development of temperature profiles in a convectively cooled rectangular fin. In addition, a corrected analytical form that reduces the 2-D problem to a 1-D one is developed. Results prove that the numerical technique is simple and fast, while the 1-D reduced analytical solution is valid only for early time, small Biot number, and thin fins.

INTRODUCTION

An extended surface is a solid that experiences energy transfer by conduction within its boundaries, as well as energy transfer by convection (and/or radiation) between its boundaries and the surroundings. Although there are many different situations that involve combined conduction-convection effects, the most frequent application is one in which an extended surface is used specifically to enhance the heat transfer rate between a solid and an adjoining fluid (fins). The applications of such extended surfaces is a common practice in the fields of thermal and electronic engineering.

The steady state temperature profiles of fins with different

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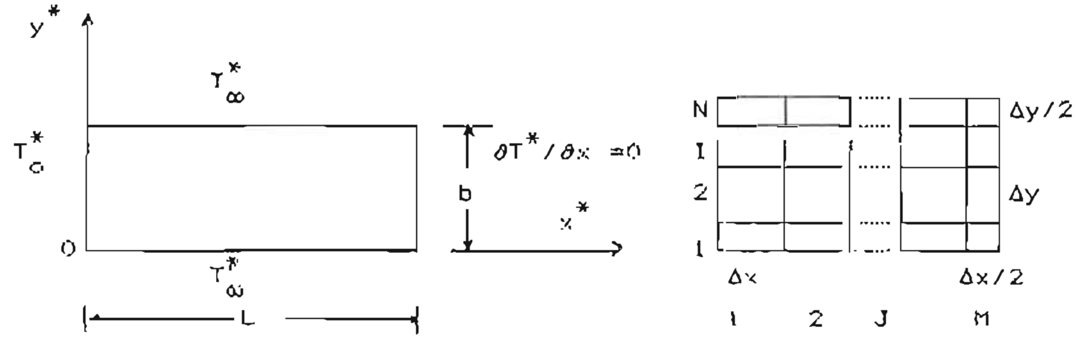
geometries and different boundary conditions has been studied by many investigators. Analytical solutions for some steady state cases are reported in [1,2]. The heat conduction problem in a semi-infinite region with temperature-dependent material properties has been dealt with in [3].
Transient heat conduction for some simple geometries and initial and boundary conditions has been investigated in [1-4]. There has been relatively little research on transient conduction in a 2-D fin, which is convectively cooled and its base is subjected to a sudden step temperature change. The analytical solution of this problem with simpler boundary conditions is given in [1]. This solution is very tedious and complicated. As reported in [4], an analytical solution of the considered problem has been obtained by Chu et al.,1982. The obtained solution is very tedious and converges very slowly. Yi-Hsu et al.[4] developed an analytical method that reduces the problem of transient heat conduction in a 2-D rectangular fin to that of a 1-D problem -Fig.1. According to their solution procedure the following inaccurate solution was obtained for a rectangular fin whose thickness is much smaller than its length, i.e $\varepsilon << 1$:

$$\overline{T}(x,t) = \sum_{j=1}^{\infty} \frac{8}{(4j-1)\pi} \left\{ \exp(-\lambda_{2j}t) + \frac{D}{\lambda_{2j}} \left[1 - \exp(-\lambda_{2j}t) \right] \right\} \sin(\lambda_{2j}\, x) \qquad (1)$$

where $\overline{T} = \int_0^1 T(x,y,t)dy$ is the average dimensionless temperature

$$\lambda_j = \left[\frac{(2j-1)\pi}{2}\right]^2 - D$$

$$D = \frac{12(H^2+2H)}{(12+8H+H^2)}, \qquad \text{and} \qquad H = hb/k$$



(a) Schematic diagram (b) Mesh design

Fig. 1 Two-dimensional rectangular fin

This work is motivated by the inaccuracy noticed in the temperature profile described by Eq.(31) in the solution presented in [4]. The temperature profile is reproduced in Eq.(1) for convenience. A careful review of the the analysis resulted in a corrected form which is presented later in this work. To confirm the corrections a finite difference scheme is developed for the same problem. Both analysis and numeric solutions are compared for the steady state heat conduction as well as the transient case.

PROBLEM FORMULATION AND SOLUTION

The specified problem is the transient heat conduction in a 2-D rectangular fin shown in Fig.1. The following assumptions are applied:

1- constant thermophysical material properties
2- constant convective heat transfer coefficient
3- the tip surface of the fin is adiabatic
4- no internal heat generation

The heat conduction equation for the fin is then given by

$$\rho c \frac{\partial T^*}{\partial t^*} - k \frac{\partial^2 T^*}{\partial x^{*2}} - k \frac{\partial^2 T^*}{\partial y^{*2}} = 0 \tag{2}$$

and the boundary conditions are:

$$- k \frac{\partial T^*}{\partial y^*}\Big]_{y^*=0} = h\ (T^*_\infty - T^*\big|_{y^*=0})$$

$$- k \frac{\partial T^*}{\partial y^*}\Big]_{y^*=b} = h\ (T^*\big|_{y^*=b} - T^*_\infty)$$

$$\partial T^*/\partial x^*\big|_{x^*=L} = 0$$

$$T^*\big|_{x^*=0} = T^*_o \tag{3}$$

The initial condition is:

$$T^*\big|_{t^*=0} = T^*_\infty \tag{4}$$

Defining the following dimensionless quantities:

$$x = x^*/L, \quad y = y^*/b, \quad \varepsilon = b/L, \quad T = \frac{T^* - T^*_o}{T^*_\infty - T^*_o}, \quad t = \frac{k\ t^*}{\rho\ c\ L} = Fo$$

equations (2) and (3) can be transformed into the following dimensionless form:

$$\frac{\partial T}{\partial t} - \frac{\partial^2 T}{\partial x^2} - \frac{1}{\varepsilon^2}\frac{\partial^2 T}{\partial y^2} = 0 \tag{5}$$

The boundary conditions are:

$$- \frac{\partial T}{\partial y}\Big]_{y=0} = H\ (1 - T\big|_{y=0}) = \varepsilon\ Bi\ (1 - T\big|_{y=0}) \tag{6a}$$

$$- \frac{\partial T}{\partial y}\Big]_{y=1} = H\ (T\big|_{y=1} - 1) = \varepsilon\ Bi\ (T\big|_{y=1} - 1) \tag{6b}$$

$$\partial T / \partial x \big|_{x=1} = 0 \tag{6c}$$

$$T \big|_{x=0} = 0 \tag{6d}$$

The initial condition is:

$$T \big|_{t=0} = 1 \tag{7}$$

Solution of Eq.5 is obtained using two different techniques:
1- an approximate 1-D analysis similar to that presented in [4] and described by the inaccurate temperature profile given by Eq.(1),
2- the finite difference numerical technique as described below.
In the finite difference technique, the region of interest is descretized to M and N nodal points in the x and y directions -Fig.1b. Equation (5) is then applied to each nodal point and the dimensionless temperature of the nodal point i,j is obtained in the following form :

$$T^{t+\Delta t}_{i,j} = a_1 T^{t}_{i,j-1} + a_2 T^{t}_{i,j+1} + a_3 T^{t}_{i-1,j} + a_4 T^{t}_{i+1,j} + a_5 T^{t}_{i,j} \tag{8}$$

where the time interval $\Delta t = \Delta Fo = \dfrac{k\ \Delta t^*}{\rho c L^2}$.

The coefficients a_1, a_2, a_3, a_4, and a_5 which satisfy the boundary conditions (6) are listed in the table below, where

$A = \Delta Fo\ M^2$, $B = \Delta Fo\ (N/\varepsilon)^2$, and $C = 2(N/\varepsilon)\ \Delta Fo Bi$

Node	a_1	a_2	a_3	a_4	a_5
Interior	A	A	B	B	$1. - a_1 - a_2 - a_3 - a_4$
Row I=1	A	A	C	2B	
Row I=N	A	A	2B	C	
Column J=1	2A	A	B	B	
Column J=M	2A	0.0	B	B	
Corner 1,1	2A	A	C	2B	
Corner 1,M	2A	0.0	C	2B	
Corner N,1	2A	A	2B	C	
Corner N,M	2A	0.0	2B	C	

The obtained set of algebraic equations are then solved using the simple explicit scheme [2]. Applying the boundary condition of Eq.6b, one finds that the solution is stable if the following condition is satisfied :

$$\Delta t = \Delta Fo \le \frac{1}{2(N^2 + Bi(M/\varepsilon) + (M/\varepsilon)^2)}$$

In order to examine the validity of both solutions, the steady state 2-D analytical solution for the considered case is derived and presented in the appendix. Following the used notations, the dimensionless steady-state solution is then given by :

$$1-T = \sum_{n=1}^{\infty} C_n \, e^{2\zeta_n x/\varepsilon} \left[1 + e^{4\zeta_n (1-x)/\varepsilon} \right] \cos(\zeta_n y) \qquad (9)$$

where $C_n = \left[\frac{2 \sin \zeta_n}{\zeta_n + \sin \zeta_n \cos \zeta_n} \right] \cdot \frac{1}{(1 + e^{4\zeta_n/\varepsilon})}$

and the eigenvalues of ζ_n are given by :

$$\zeta_n \tan \zeta_n = hb/k = Bi.\varepsilon/2$$

RESULTS AND DISCUSSION

From the dimensionless Eq.8 it can be seen that the proposed finite difference technique is simple and very fast even for large time t.

Figure 2 illustrates the steady state temperature profiles as obtained from both the analytical steady-state solution (Eq.9) and the numerical solution (Eq.8) at a large time, t ≥ 1. Both profiles agree closely for any Biot number, which proves the validity of the numerical solution. Figure 3 shows the effect of Biot number on the temperature profile for steady-state condition. The time development of temperature profiles as obtained from the numerical solution (Eq.8) are plotted on Fig.4. These results agree very good with those supposed to be obtained by Yi-Hsu et al. using Eq.1 for the same conditions- Fig 5. The effect of Biot number on the transient temperature profile at time 0.01 is plotted on Fig.6 as obtained numerically using Eq.8. The numerical solution shows some difference between the center and surface temperature profiles for Bi > 1 as illustrated on Fig.7.

Eq.1 is tested in this work. Unfortunately this solution in the presented form fails to give any of the presented results neither here nor in the concerned work of Yi-Hsu et al.[4]. Following the same procedure used in [4] an attempt is made to correct Eq.1. The corrected version of Eq.1 is then obtained in the form :

$$\overline{T}(x,t) = \sum_{j=1}^{\infty} \frac{4}{(2j-1)\pi} \left\{ \exp(-\lambda_j t) - \frac{D}{\lambda_j} \left[1 - \exp(-\lambda_j t) \right] \right\} \sin(\omega_j x) \qquad (10)$$

where $\omega_j = \frac{(2j-1)\pi}{2}$, and $\lambda_j = \left[\frac{(2j-1)\pi}{2} \right]^2 - D$

Figures 8 and 9 illustrate a comparison between the obtained temperature profiles for Bi=0.1 and Bi=2as obtained from numerical solution (Eq.8) and the corrected 1-D analytical solution (Eq.10). Figure 8 indicates that both solutions agree only for time t = 0.01 , while there is some deviation for t = 0.1 if Bi = 0.1. For time Bi = 2.0 both solutions completely disagree even if t = 0.1 as shown in Fig.9. This last observation may be justified by the fact that the assumption used in [4] to reduce the dimensionality of the problem lacks to represent the actual temperature variation as time progresses.

CONCLUSIONS

From the above discussion it can be deduced that the proposed simple dimensionless finite difference technique correctly predicts the time development of temperature profiles for transient conduction in a 2-D rectangular fin. In addition a corrected form is obtained by which the 2-D problem is reduced to 1-D one. This form is valid only for small time, small Biot number, and for a thin rectangular fin.

In a subsequent paper the authors present a complete analytical solution for the transient heat conduction in a 2-D rectangular fin.

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NOMENCLATURE

- b fin thickness
- c specific heat
- h convective heat transfer coefficient
- k thermal conductivity
- L fin length
- t^* real time
- t dimensionless time
- T^* temperature
- T dimensionless temperature
- T^*_∞ ambient temperature
- T^*_o temperature at the fin base
- x^* longitudinal coordinate
- x dimensionless x^*
- y^* transverse direction
- y dimensionless y^*

Greek symbols

- $\varepsilon = b/L$
- ρ density
- λ eigenvalue
- ζ eigenvalue

Dimensionless groups

- Bi Biot number $= hL/k$
- Fo Fourier number $= k\,t^*/(\rho c L^2)$

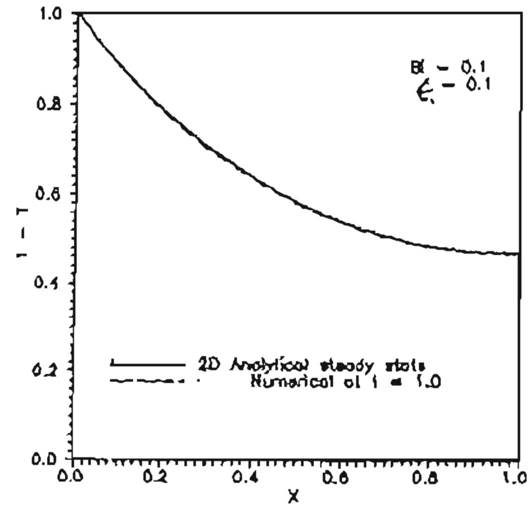


Fig.(2) Steady-state temperature profile (Bi=0.1)

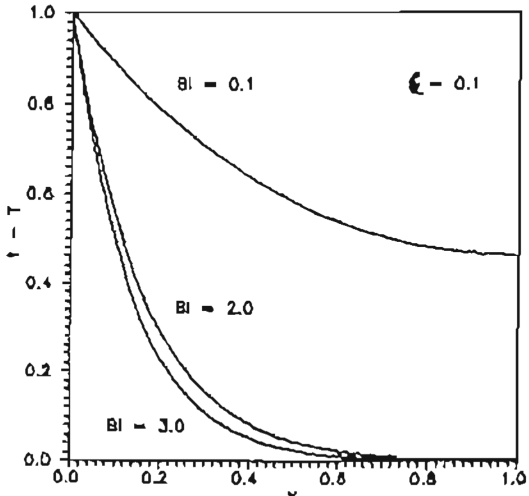


Fig.(3) Effect of Bi on steady-state temperature profiles

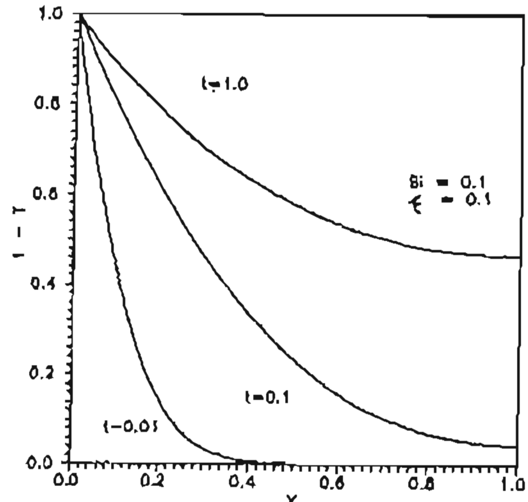


Fig.(4) Time development of temperature profiles

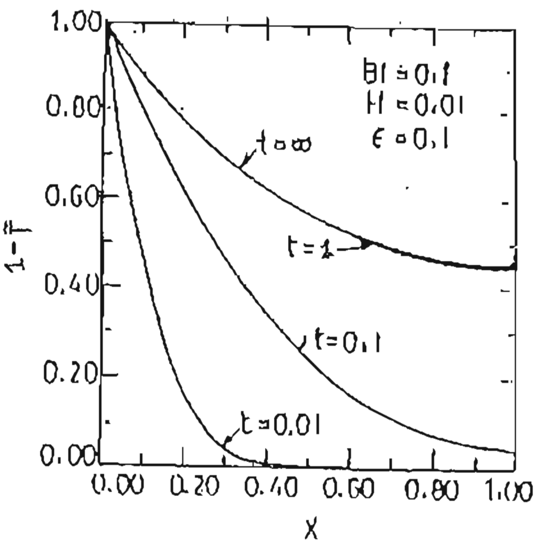


Fig.(5) Time development of temperature profiles (Reproduced from [4])

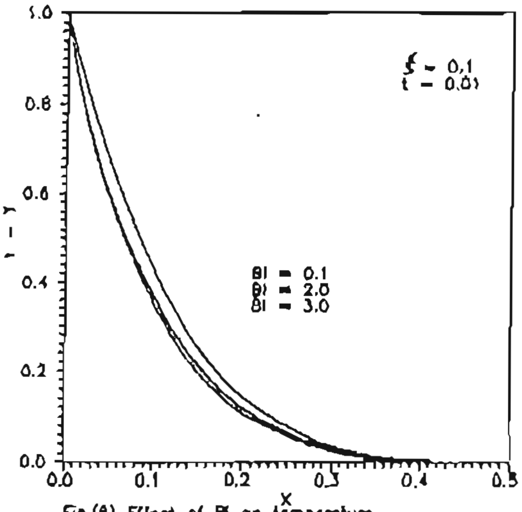


Fig.(6) Effect of Bi on temperature profiles at t = 0.01

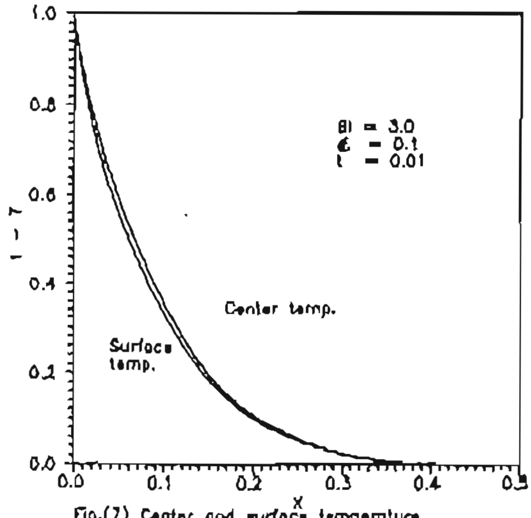


Fig.(7) Center and surface temperature profiles

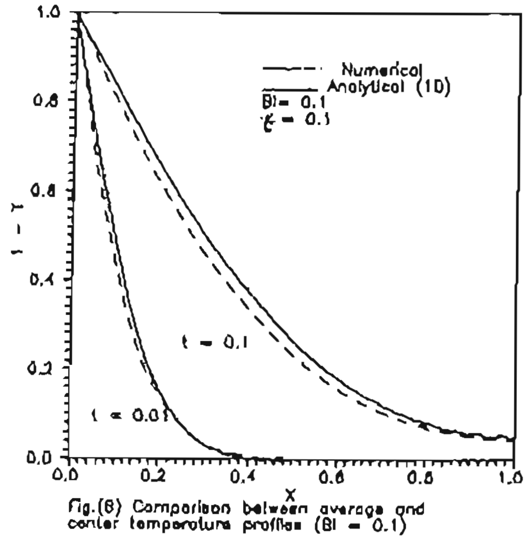


Fig.(8) Comparison between average and center temperature profiles (Bi = 0.1)

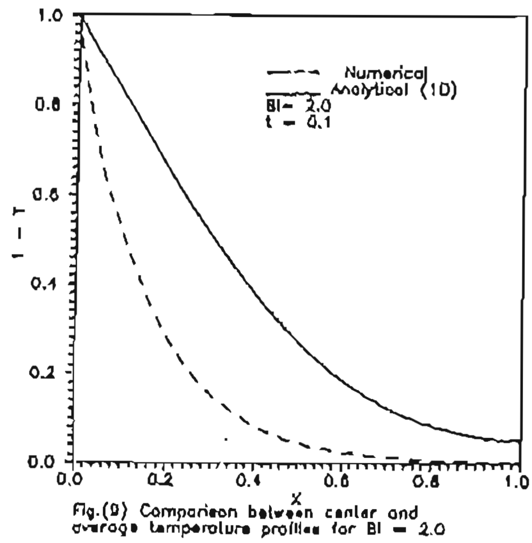


Fig.(9) Comparison between center and average temperature profiles for Bi = 2.0

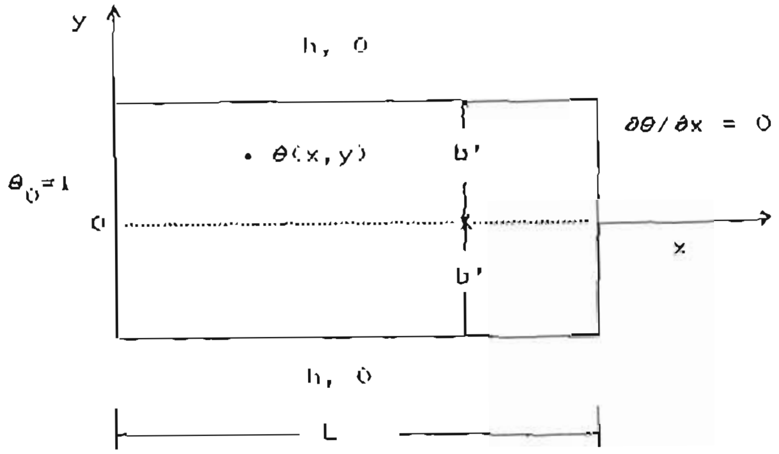
APPENDIX

Solution of 2-D steady-state heat conduction in rectangular fin

The original heat conduction equation is given by :

$$\frac{\partial^2\theta}{\partial x^2} + \frac{1}{c'^2}\frac{\partial^2\theta}{\partial y^2} = 0 \qquad (1)$$

where $\theta = \dfrac{T^* - T^*_\infty}{T^*_0 - T^*_\infty}$, and $c' = b'/L$



Referring to the above figure the boundary conditions are:

$$\theta(0,y) = 1 = \theta_0, \qquad \frac{\partial\theta(1,y)}{\partial x} = 0, \qquad \frac{\partial\theta(x,0)}{\partial y} = 0, \quad \text{and}$$

$$-\frac{\partial\theta(x,1)}{\partial y} = \frac{hb'}{k}\,\theta(x,1)$$

Let $\theta = Y(y).X(x)$ (2)

Substituting in Eq.1 and separating the variables, we get

$$\frac{d^2Y(y)}{dy^2} + c'^2\lambda^2 Y = 0 \qquad (3)$$

with $\dfrac{dY(0)}{dy} = 0$, and $-\dfrac{dY(1)}{dy} = \dfrac{hb'}{k}\,Y(1)$

and

$$\frac{d^2X(x)}{dx^2} - \lambda^2 X = 0 \qquad (4)$$

with $X(0) = 0$, and $\dfrac{dX(1)}{dx} = 0$

Solution of Eq.3 is given by:

$$Y(y) = C_n\,\phi_n(y) \qquad (5)$$

where $\phi_n(y) = \cos(c'\lambda_n y)$

and the eigenvalues of λ_n are positive roots of the transcendental equation $\zeta_n \tan\zeta_n = hb'/k$

where $$\zeta_n = c' \lambda_n$$
Solution of Eq.4 is then given by:

$$X(x) = A\, e^{\lambda_n x} + B\, e^{-\lambda_n x} \tag{6}$$

Applying the corresponding boundary conditions, Eq.6 becomes

$$X(x) = A\,(e^{\lambda_n x} + e^{2\lambda_n}\, e^{-\lambda_n x}) \tag{7}$$

Substituting Eqs.5 and 7 into Eq.2, we get

$$\theta(x,y) = \sum_{n=1}^{\infty} a_n\, (e^{\lambda x} + e^{2\lambda}\, e^{-\lambda x})\cos \zeta_n y \tag{8}$$

where $a_n = C_n A$. From Eq.8, we obtain

$$\theta_o = 1 = \sum_{n=1}^{\infty} a_n\, (1 + e^{2\lambda_n})\cos\zeta_n y \tag{9}$$

Using the property of orthogonality [1], the coefficients a_n may be calculated by the expression:

$$a_n = \left[\frac{2\sin\zeta_n}{\zeta_n + \sin\zeta_n \cos\zeta_n}\right] \frac{1}{(1 + e^{2\lambda_n})} \tag{10}$$