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ON THEORY OF THE INVERSE PROBLEM OF STEADY, TWO-DIMENSIONAL HEAT CONDUCTION IN A Hollow CYLINDER WALL

"حل نظرى للمسألة المعكوسة لانتقال حرارى مستقر فى اسطوانة مجوفة"

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ملخص : تم ايجاد حل نظرى لمسألة معكوسة لانتقال حرارى مستقر فى اتجاهين خلال حائط اسطوانة مجوفة. الحل التحليلى الناتج فى صورة متتالية متقاربة . مسألة لها حل تحليلى معلوم استخدمت لفحص شرعية طريقة الحل الجديدة . النتائج أوضحت أنه باستخدام عدد محدود فى متتالية الحل فان الحلول التقريبية الناتجة ذات دقة عالية وهذا اتضح من المقارنة مع حلول تحليلية معلومة .

Abstract

An inverse problem of steady, two-dimensional heat conduction in a hollow cylinder wall of constant thermal conductivity, has been analyzed. Solution in form of a convergent series has been obtained. Simple test problem, has known exact solution, proves validity of the solution method. By truncating the series, approximate solutions of simple form result which are of reasonable accuracy and compare well with known exact solutions.

1. Introduction

In many physical situations, the heat transfer characteristics at one side of a domain have to be evaluated from corresponding measurements at the opposite side without auxiliary information at the other sides of the body. This problem is identified as inverse heat conduction problem (IHCP) [2] and is distinctly different from the direct problem, in which the temperature distribution of a body is to be determined from data specified over the entire surface. In practice, such direct heat transfer problems occur mainly in design applications while inverse problems are encountered in analysis of experimental data. The inverse problem arises when a surface may be unsuitable for fixation of temperature sensor due to technical difficulty, or when the accuracy of the surface measurement may seriously impaired by the presence of the sensor, which may affect the surface condition as well as disturb the flow and heat transfer close to the surface. Therefore, it is desired in some situations to predict the temperature and heat flux at a certain surface from data measured at the opposite side surface only. Generally, the inverse problems in heat conduction are divided to steady-state and transient problems [4]. The inverse problem of transient, 1-d. heat conduction has analytically been solved for sample geometries at first by Burggraf [6] and later by Widder [5].

In the present paper, we analyze an inverse problem type for steady, 2-dimensional heat conduction in a hollow cylinder wall of constant thermal conductivity. The problem is characterized by specifying temperature and its radial derivative profiles at one boundary surface; each profile is a continuous and deferential function of the variable z (cf. Fig. 1). The objective of the present work is to obtain analytically the (r,z) solution of temperature and heat flux in the cylinder wall including the boundaries. The main difficultly of the stated problem lies in the fact that only two boundary conditions are known and at one side surface. The problem is quite different from the corresponding direct one whose solution necessitates four boundary conditions (temperature or heat flux or thereof); with two for each coordinate.

Analytical 2-dimensional solution of the temperature and heat flux for the hollow cylinder wall has been obtained, which in the form of a convergent series. The solution is somewhat similar to that derived by Burggraf [6] for the inverse problem of transient, one-dimensional heat conduction. The similarity between the two solution lies mainly in that the z-space variable in our solution simulates the role of the time variable in the transient, 1-d. solution.

The present analytical solution may be one of considerable practical interest, however, to some experimental heat transfer investigations. The method may be applied to evaluate measured data from a steady-state experiment, in which heat flux profile is measured at an isothermal surface, or temperature profile is measured at perfectly insulated surface; and it is required to estimate the temperature and heat flux distributions at the opposite surface.

2. Problem description and solution

Figure 1 states an inverse problem type for steady, two-dimensional, heat conduction in a hollow cylinder of constant thermal conductivity. The temperature and its exterior radial derivative at the outside surface are known as functions of the coordinate z. Our main purpose is to obtain an expression for the (r,z) temperature field of the cylindrical wall domain including the surface boundaries.

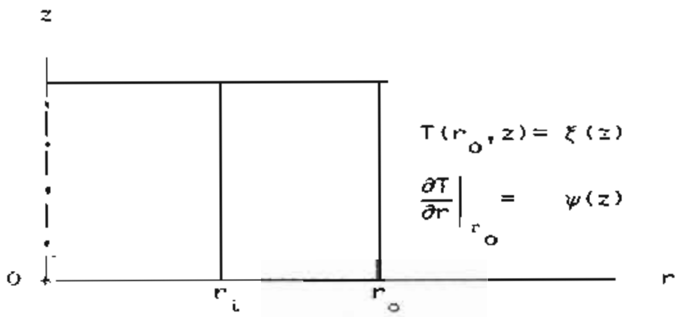

Figure 1 Problem description

If there is no heat generation, this problem may be modeled by the governing differential equation,

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0, \tag{1}$$

with the boundary conditions,

$$T(r_o,z) = \xi(z), \tag{2}$$

$$\left.\frac{\partial T}{\partial r}\right|_{r_o} = -\frac{q_r(r_o,z)}{k} = \psi(z). \tag{3}$$

Similar to that followed in analysis of the corresponding problem in the case of a planar, two-dimensional wall [7-8], the general solution of the two dimensional temperature field can be assumed to be an infinite series involving the exterior radial temperature gradient as

$$T(r,z) = \sum_{n=0}^{\infty} \phi_n(r) \left.\frac{\partial^n T}{dr^n}\right|_{r_o} \tag{4}$$

After some mathematical manipulations using the basic differential equation (1) with equation (4), the above expression of the temperature field can be transformed to express by

$$T(r,z) = \sum_{n=0}^{\infty} A_n(r) \frac{d^{2n}T_o}{dz^{2n}} + \frac{1}{k}\sum_{n=0}^{\infty} B_n(r) \frac{d^{2n}q_o}{dz^{2n}} \tag{5}$$

This mathematical transformation process is simillar to that made for the plane wall [8]. For simplify notion , we set $T(r_o,z) = T_o$ and $q_r(r_o,z) = q_o$ in the above equation as well as in the remainder of the paper. It is important to observe from the r.h.s. of equation (5) that for an perfectly insulated surface at r_o, the terms of $A_n(r)$-coefficients involving in the first summation sign satisfies alone the solution, while for (isothermal) surface condition at the outer surface, the terms of $B_n(r)$-coefficients involving in the second summation sign satisfies alone the solution. Now, the remaining problem is to determine the functions $A_n(r)$ and $B_n(r)$ in eq. (5). These functions are determined by substituting equation (5) into the basic differential equation (1), this yields

$$\sum_{n=0}^{\infty} \left[rA_{n-1}(r) + A_n'(r) + rA_n''(r) \right] \frac{d^{2n}T_o}{dz^{2n}} +$$

$$\frac{1}{k}\sum_{n=0}^{\infty} \left[rB_{n-1}(r) + B_n'(r) + rB_n''(r) \right] \frac{d^{2n}q_o}{dz^{2n}} = 0. \tag{6}$$

A solution is obtained by requiring that each term inside the brackets of equation (5) is identically zero, thus one obtains

$$[A_o'(r) + rA_o''(r)] = 0; \quad rA_{n-1}(r) + A_n'(r) + rA_n''(r) = 0 \tag{7}$$

$$[B_o'(r) + rB_o''(r)] = 0; \quad rB_{n-1}(r) + B_n'(r) + rB_n''(r) = 0 \tag{8}$$

With n =1,2,3,..in the above two equations. The boundary conditions on the $A_n(x)$ and $B_n(x)$ functions are determined from the requirement that the problem solution exactly matches the two known boundary conditions at the outer surface. The first boundary condition (eq. (2)) fulfills the general equation (5) so that

$$T(r_o,z) = T_o = \sum_{n=0}^{\infty} A_n(r_o) \frac{d^{2n}T_o}{dz^{2n}} + \frac{1}{k}\sum_{n=0}^{\infty} B_n(r_o) \frac{d^{2n}q_o}{dz^{2n}} \tag{9}$$

This condition gives

$$A_o(r_o) = 1, \; B_o(r_o) = 0 \; \text{ and } \; A_n(r_o) = B_n(r_o) = 0; \; n = 1,2,.. \tag{10}$$

The second boundary condition (eq. (3)) satisfies the general solution that

$$q_o = -k\frac{\partial T}{\partial r}\bigg|_{r_o} = -k\sum_{n=0}^{\infty} A_n'(r_o)\frac{d^{2n}T_o}{dz^{2n}} - \sum_{n=0}^{\infty} B_n'(r_o)\frac{d^{2n}q_o}{dz^{2n}} \tag{11}$$

which gives

$$B_o'(r_o) = -1, \; A_o'(r_o) = 0 \; \text{ and } \; A_n'(r_o) = B_n'(r_o) = 0; \; n = 1,2,.... \tag{12}$$

The solution to eqs. (7)&(8) subject to the boundary conditions given by eqs. (10) and (12) completely determines the $A_n(r)$ and $B_n(r)$ functions. Note that these functions must be determined in a sequential manner starting with $A_o(r)$ and $B_o(r)$. General results are not available as in the case of plane wall [7], however, the leading terms are given by

$$\left.\begin{aligned}
&A_o(r) = 1, \qquad A_1(r) = \frac{r_o^2}{4}\left[1 - \left(\frac{r}{r_o}\right)^2 + 2\ln\left(\frac{r}{r_o}\right)\right],\\
&A_2(r) = -\frac{r_o^4}{16}\left[\frac{5}{4} - \left(\frac{r}{r_o}\right)^2 - \frac{1}{4}\left(\frac{r}{r_o}\right)^4 + \left(1 + 2\left(\frac{r}{r_o}\right)^2\right)\ln\left(\frac{r}{r_o}\right)\right],\\
&A_3(r) = \frac{r_o^6}{64}\left[\frac{10}{36} + \frac{1}{4}\left(\frac{r}{r_o}\right)^2 - \frac{1}{2}\left(\frac{r}{r_o}\right)^4 - \frac{1}{36}\left(\frac{r}{r_o}\right)^6 + \left(\frac{1}{6} + \left(\frac{r}{r_o}\right)^2 + \frac{1}{2}\left(\frac{r}{r_o}\right)^4\right)\ln\left(\frac{r}{r_o}\right)\right],
\end{aligned}\right\} \tag{13}$$

and the solution for $B_n(r)$ are:

$$B_o(r) = -r_o \ln(\frac{r}{r_o}), \quad B_1(r) = \frac{r_o^3}{4}\left[1 - (\frac{r}{r_o})^2 + (1 + (\frac{r}{r_o})^2) \ln(\frac{r}{r_o})\right],$$

$$B_2(r) = -\frac{r_o^5}{64}\left[\frac{3}{2} - \frac{3}{2}(\frac{r}{r_o})^4 + (1 + 4(\frac{r}{r_o})^2 + (\frac{r}{r_o})^4) \ln(\frac{r}{r_o})\right], \tag{14}$$

$$B_3(r) = \frac{r_o^7}{128}\left[\frac{11}{108} + \frac{1}{4}(\frac{r}{r_o})^2 - \frac{1}{4}(\frac{r}{r_o})^4 - \frac{11}{108}(\frac{r}{r_o})^6 + (\frac{1}{18} + \frac{1}{2}(\frac{r}{r_o})^2 + \frac{1}{2}(\frac{r}{r_o})^4 + \frac{1}{18}(\frac{r}{r_o})^6) \ln(\frac{r}{r_o})\right],$$

Successive terms are generated easily from equations (7) and (8).

By substituting $A_n(r)$-and $B_n(r)$-series from eqs. (13)&(14) into equation (5), the general solution of temperature field is determined which can also be expressed as follows

$$T(r,z) = \left[T_o - \frac{r_o}{K}\ln(\frac{r}{r_o})q_o\right] + \sum_{n=1}^{\infty} A_n(r)\frac{d^{2n}T_o}{dz^{2n}} + \frac{1}{k}\sum_{n=1}^{\infty} B_n(r)\frac{d^{2n}q_o}{dz^{2n}} \tag{15}$$

From the right-hand side of the above equation, it is important to note that the term inside the brackets represents a steady, one-dimensional heat conduction solution (in r-direction) for constant T_o and q_o values; and the effect of two-dimensional heat flow are included in the remaining terms.

Finally, from equation (15) with Fourier 's law

$$q_r(z) = -k \frac{\partial T}{\partial r} \tag{16}$$

the radial heat flux can be expressed by

$$q_r(z) = \frac{r_o}{r}q_o - k\sum_{n=1}^{\infty} A_n'(r)\frac{d^{2n}T_o}{dz^{2n}} - \sum_{n=1}^{\infty} B_n'(r)\frac{d^{2n}q_o}{dz^{2n}} \tag{17}$$

Temperature and heat flux of the inner cylinder surface can be calculated from eq. (15) and eq. (17), respectively; with $r = r_i$.

It is evident that the solution is explicit. The basic requirement of the analysis is that the surface temperature and heat flux are assumed to be uniform and varied with wall length. In other words, the two functions $\psi(z)$and $\xi(z)$ must be continuous and its first n derivatives should be exist. Subject to this condition the method is applicable.

3. Test Problem and Discussion

In the preceding section, general solution of the stated inverse problem of steady heat conduction in a hollow cylinder wall has been obtained. The axial distribution for each of the temperature and its exterior radial-gradient – – (both at at the outer surface r_o) are the two boundary conditions required to obtain the present solution. Each of the two b. conditions has to be in form of a continuous and differential function. However, it is important to consider a test problem, has known exact solution, in order to examine the validity of the present solution method as well as to illustrate its application in more detail. As no a such test problem is available to us from references, we construct the following one :

Consider a hollow cylinder (r_i, r_o, L) of constant thermal conductivity exposed to constant heating flux q at outer surface. The temperature of the inner surface is $T(r_i, z) = A_o + A_1 \cos \frac{\pi z}{L}$. The other two boundary surface (r,0) and (r,L) are insulated. This problem has analytically been solved by the method of superposition principle in appendix (A). This solution reads

$$T(r,z) = A_o + A_1 \cos(\frac{\pi z}{L}) \left[\frac{I_o(\frac{\pi}{L} r) K_1(\frac{\pi}{L} r_o) + I_1(\frac{\pi}{L} r_o) K_o(\frac{\pi}{L} r)}{I_o(\frac{\pi}{L} r_i) K_1(\frac{\pi}{L} r_o) + I_1(\frac{\pi}{L} r_o) K_o(\frac{\pi}{L} r_i)} \right] - \frac{q r_o}{k} \ln(\frac{r}{r_i}) \quad (18)$$

Now, we come to use the present method to solve the same problem, however, using only the following two b. conditions at the outer surface;

$$q_o = -q, \quad \text{(given in the original problem)} \quad (19)$$

$$T_o = T(r_o, z) \quad \text{(calculated from eq. (18) with } r = r_o) \quad (20)$$

As q_o is constant, the general solution (eq. (15) in sec. 2) is reduced to

$$T(r,z) = \left[T_o + \frac{r_o}{K} \ln(\frac{r}{r_o}) q_o \right] + \left[\sum_{n=0}^{\infty} A_n(r) \frac{d^{2n} T_o}{dz^{2n}} \right] \quad (21)$$

The temperature derivative terms in the above equation is calculated using eq. (20), and $A_n(r)$ from eq. (13). Thus T(r,z) becomes known. As no closed form solution seems to be found by eq. (21), only results in digital values are available. Therefore, to check the validity of this solution with referred to the known exact one, the temperature profile is calculated at the inner surface by eq. (21) with $r = r_i$. Figure 1 shows the resultant comparison, where results from eq. (21) (with n = 1, and n = 2) are compared with the exact solution. The results are represented in terms of the dimensionless inner surface temperature $(T(r_i, z)/(A_o + A_1))$ versus the axial position (z/L). It is seen that the present solution converges very rapidly, the results for n = 2 being indistinguishable from those of exact solution. The maximum absolute error in the approximate solution with $n = 1$ is in order of magnitude of 1%, whereas it is fast 0% for $n = 2$, see Figure 1b.

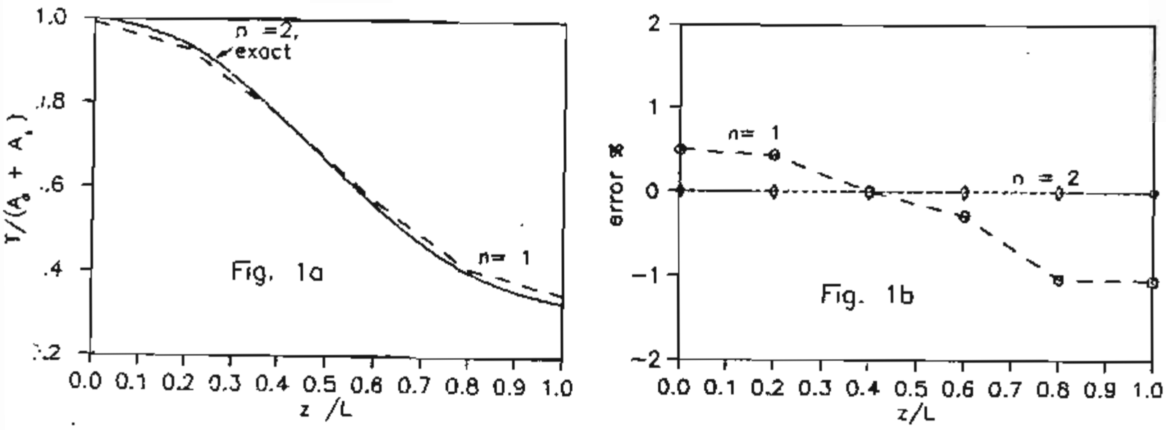

Figure 1 Comparison exact solution with the present solution (for $r_i = 1$ cm, $L/r_o = 4$, $r_o/r_i = 2$, $k = .14$ W/(m^{2}C), $q = 30$ W/cm^2)

4. Final remarks

So far, the 2-dimensional temperature solution of an inverse problem of steady heat conduction in a hollow cylinder of constant thermal conductivity, has been obtained. The solution is explicit

The prerequisite is that the temperature and its r-gradient are known at the boundary surface r_o; both are functions of the z variable. These two functions must be continues and differential. Subject to this condition the present method is applicable.

It is important to note that in carrying out the analysis, no reference was made to the boundary conditions on the two boundary planes (r,0) and (r,L). However, this omission is no cause for concern. Because of the known smooth nature of the linear governing equation, the temperature distribution T(r,0) and T(r,L) is uniquely specified when the surface temperature profile $T(r_o,z)$ and its r-gradient are given over the interval $(0 \le z \le L)$. The test problem, presented in section 3, confirm this fact and reveal that the present method is correct and reliable. Results show that representation of the solution by a few terms of the series is appropriate and of reasonable accuracy.

The solution may also be one of considerable practical interest, however, to some experimental heat transfer investigations in pipe, as for a steady experiment, in which heat flux distribution is measured at an isothermal outer surface, or temperature profile is measured along an insulated outer surface, and it is desired to predict the corresponding values of temperature and heat flux at the opposite side surface. However, in such practical situations, the data are not available in form of convenient theoretical expressions for temperature or heat flux but as tabulated data measured at discrete points. Therefore, to apply the present solution, the data should be expressed analytically, by curve-fit formulas (e.g., polynomial) using (for instance) the least squares technique, in order to evaluate the derivatives in equation (15).

Appendix (A)

In this appendix we construct an exact analytical solution for a particular problem of steady two-dimensional heat conduction in a hollow cylinder. This analytical solution will be used as a test to make check on the validity of the general solution method derived in this paper in section 2. For this purpose, we consider a hollow cylinder of constant thermal conductivity. The temperature at the inside surface is given as a cosine function, and the outer surface is exposed to constant heating flux q, while the other sides are insulated. According to the reference coordinate system depicted in Figure 1, the problem can be modeled by the governing differential equation

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0, \qquad \text{(A1)}$$

with the boundary conditions:

$$\left.\frac{\partial T}{\partial r}\right|_{r_o} = -\frac{q}{k} \quad \text{(A2)}, \qquad T(r_i, z) = A_o + A_1 \cos\frac{\pi z}{L} \quad \text{(A3)}$$

$$\left.\frac{\partial T}{\partial z}\right|_{o} = 0 \quad \text{(A4)}, \qquad \left.\frac{\partial T}{\partial z}\right|_{L} = 0 \quad \text{(A5)}$$

where A_o and A_1 are known constants of temperature units.

The present problem is of a direct type, since it has 4 boundary conditions; two in z-direction and two in r-direction. This problem can not be solved directly by employing the classical method of variables separation, since its application requires that differential equation and three of the boundary conditions are homogeneous. This prerequisite is not satisfied. Only, the governing partial differential equation (A1) and two boundary conditions (A5) and (A4) are homogeneous. However, it is possible to solve the problem by using the method of superposition. According to this principle of superposition, the solution may be assumed to be

$$T(r,z) = T_z(r,z) + T_1(r) \qquad \text{(A6)}$$

Here, $T_1(r)$ is assumed to satisfy the one-dimensional solution (in r-direction) by

$$\frac{\partial^2 T_1}{\partial r^2} + \frac{1}{r}\frac{\partial T_1}{\partial r} = 0, \qquad \text{(A7)}$$

with

$$\left.\frac{\partial T_1}{\partial r}\right|_{r_o} = -\frac{q}{k} \quad \text{(A8)}, \qquad T_1(r_i, z) = 0 \quad \text{(A9)}$$

Hence, the solution of eq. (A7) subject to the b. conditions (A8) and (A9) is

$$T_1(r) = -\frac{qr_o}{k}\ln(r/r_i) \qquad (A10)$$

Then, combining eqs. (A1)-(A9), we fined that $T_2(r,z)$ is satisfied by

$$\frac{\partial^2 T_2}{\partial r^2} + \frac{1}{r}\frac{\partial T_2}{\partial r} + \frac{\partial^2 T_2}{\partial z^2} = 0, \qquad (A11)$$

with

$$\left.\frac{\partial T_2}{\partial r}\right|_{r_o} = 0 \qquad (A12), \qquad T_2(r_i,z) = A_o + A_1\cos\frac{\pi z}{L} \qquad (A13)$$

$$\left.\frac{\partial T_2}{\partial z}\right|_{r_o} = 0 \qquad (A14), \qquad \left.\frac{\partial T_2}{\partial z}\right|_{L} = 0 \qquad (A15)$$

By use of the classical method of variables separation $T_2(r,z)$ can be assumed,

$$T_2(r,z) = R(r) \, . \, \mathbb{Z}(z) \qquad (A16)$$

When substituted into equation (A11) this yields

$$\frac{1}{R}\frac{d^2R}{dr^2} + \frac{1}{rR}\frac{dR}{dr} = -\frac{1}{\mathbb{Z}}\frac{d^2\mathbb{Z}}{dz^2} \qquad (A17)$$

The left side, a function of r alone, can equal the right side, a function of z alone, only if both sides equal a constant value, say λ^2, thus one obtains

$$\frac{d^2\mathbb{Z}}{dz^2} + \lambda^2\mathbb{Z} = 0 \qquad (A18)$$

$$\frac{dR^2}{dr^2} + \frac{1}{r}\frac{dR}{dr} - R\lambda^2 = 0 \qquad (A19)$$

The solution of equation (A17) can be written as

$$\mathbb{Z}(z) = C_1 \sin\lambda z + C_2 \cos\lambda z \qquad (A20)$$

and that of equation (A18) as

$$R(r) = B_1 I_o(\lambda r) + B_2 K_o(\lambda r) \qquad (A21)$$

Applying the boundary conditions(A14)on eq.(A20) gives $C_1 = 0$, and (A15) yields $C_2 \sin(\lambda L) = 0$ which requires that $\sin(\lambda L) = 0$ or $\lambda_n = \frac{n\pi}{L}$; n =1,2,... Thus, equation (A20) becomes

$$\mathbb{Z}(z) = \sum_{n=0}^{\infty} C_2 \cos(\lambda_n z) \qquad (A22)$$

By using the boundary conditions (A12) with eq. (A21); the b.

condition (A12) yields $B_2 = B_1 \; I_1(\lambda r_o)/K_1(\lambda r_o)$. Thus, eqution (A21) reads as

$$R(r) = B_1\left(I_o(\lambda r) + I_1(\lambda r_o)\,K_o(\lambda r)/K_1(\lambda r_o)\right) \qquad (A23)$$

Substituting eq. (A23) & (A(22) into eq. (A16), one obtains

$$T_2(r,z) = \sum_{n=0}^{\infty} C_n \cos(\lambda_n z)\left(I_o(\lambda_n r) + I_1(\lambda_n r_o)K_o(\lambda_n r)/K_1(\lambda_n r_o)\right) \qquad (A24)$$

where the constants B_1 and C_2 are combined and replaced by C_n. Finally, applying the boundary condition (A13) on equation (A24) gives

$$A_o + A_1\cos\frac{\pi z}{L} = \sum_{n=0}^{\infty} C_n\cos(\lambda_n z)\left(I_o(\lambda_n r_1) + I_1(\lambda_n r_o)\,K_o(\lambda_n r_1)/K_1(\lambda_n r_o)\right)$$

which holds if $C_2 = C_3 = \ldots = C_n = 0$ and $C_o = A_o$ and

$$C_1 = A_1 \frac{K_1(\frac{\pi}{L}r_o)}{\left\{I_o(\frac{\pi}{L}r_1)K_1(\frac{\pi}{L}r_o) + I_1(\frac{\pi}{L}r_o)K_o(\frac{\pi}{L}r_1)\right\}} \qquad (A25)$$

Thus, equations (A24) becomes

$$T_2(r,z) = A_o + A_1\cos(\frac{\pi z}{L})\left[\frac{I_o(\frac{\pi}{L}r)\;K_1(\frac{\pi}{L}r_o) + I_1(\frac{\pi}{L}r_o)\;K_o(\frac{\pi}{L}r)}{I_o(\frac{\pi}{L}r_1)\;K_1(\frac{\pi}{L}r_o) + I_1(\frac{\pi}{L}r_o)\;K_o(\frac{\pi}{L}r_1)}\right] \qquad (A26)$$

Here, substitution equations (A10) and (A26) into equation (A6) yields

$$T(r,z) = A_o + A_1\cos(\frac{\pi z}{L})\left[\frac{I_o(\frac{\pi}{L}r)K_1(\frac{\pi}{L}r_o)+I_1(\frac{\pi}{L}r_o)K_o(\frac{\pi}{L}r)}{I_o(\frac{\pi}{L}r_1)K_1(\frac{\pi}{L}r_o)+I_1(\frac{\pi}{L}r_o)K_o(\frac{\pi}{L}r_1)}\right] - \frac{qr_o}{k}\ln(\frac{r}{r_i}) \qquad (A27)$$

The above equation is the final result of the temperature solution which satisfies the given boundary conditions.
Finally, the radial heat flux can be calculated by Fourier's law;

$$q_r(z) = -k\,\frac{\partial T}{\partial r} \qquad (A28)$$

with the radial temperature gradient calculated from eq. A(27) as

$$q_r(z) = -\frac{k\pi}{L}A_1\cos(\frac{\pi z}{L})\left[\frac{I_1(\frac{\pi}{L}r)K_1(\frac{\pi}{L}r_o) - I_1(\frac{\pi}{L}r_o)K_1(\frac{\pi}{L}r)}{I_o(\frac{\pi}{L}r_1)K_1(\frac{\pi}{L}r_o)+I_1(\frac{\pi}{L}r_o)K_o(\frac{\pi}{L}r_1)}\right] + q\,(\frac{r_o}{r}) \qquad (A29)$$

Nomenclature

$A_n(r)$ & $B_n(r)$	r-dependent functions, see eqs. (13) and (14).
A_o & A_1	constants, see eq. (A3) in Appendix (A)
k	thermal conductivity.
K_o & k_1	modified Bessel Functions of the 2nd kind.
I_o & I_1	modified Bessel Functions of the 1st kind.
L	cylinder length
q_o	heat flux at the outer surface r_o, ($=q_r(r_o,z)$).
n	number of terms in the solution series
r_o	outer cylinder radius
r_i	inner cylinder radius
T	temperature.
T_o	temperature of the outer surface r_o, ($=T_o(r_o,z)$.
r, z	cylindricall coordinates.
$\psi(z)$ & $\xi(z)$	z-dependent functions, see eqs. (2) and (3).
$\phi(z)$	z-dependent functions, see eq. (4).

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