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FREE VIBRATIONS OF CYLINDRICAL SHELLS BY THE NODAL LINE FINITE DIFFERENCE METHOD-PART I-THEORY

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الإهتزاز الحر للقشريات الإسطوانية باستخدام طريقة الفروق المحددة لخطوط التقسيم-الجزء الأول- النظرية

الخلاصة

حيث أن الجهد المطلوب فى التحليل الديناميكى للمنشآت عدديا يتأثر بعدد درجات الحرية والتى يترتب علـــى زيادتـــها زيادة احتمالات تراكم الأخطاء وجهد الحساب - يتناول هذا البحث كيفية تطبيق طريقة الفروق المحددة لخطوط التقســيم للتحليل الديناميكى لتميزها بقلة عدد درجات الحرية، وفى هذه الطريقة يتم تحويل الأنظمة المستمرة إلى أنظمة محــــددة بعدد درجات حرية مساوية لثلاث أضعاف عدد خطوط التقسيم. ويعرض البحث تطوير هذه الطريقة لتنـــاول التحليـل الإنشائى للإهتزاز الحر وتكوين المعا.لات اللازمة للحصول على الترددات والأشكال الطبيعية للإهتزاز، ومن ثم يمكننـا تطوير هذه الطريقة لحل المشاكل الديناميكية للمنشآت تحت تأثير الأحمال الديناميكية المختلفة.

1. INTRODUCTION

In this paper, the nodal line finite difference method (NLFDM) is extended to the analysis of a thin circular cylindrical shell, as shown in Fig. 1, undergoing free vibrations [1]. The material of the circular cylindrical shell is assumed to be linear elastic isotropic.

The complexity of a theoretical analysis of a vibration problem depends largely on the number of degrees of freedom of the structural system in question. Since a thin circular cylindrical shell is a continuous system it has an infinite number of degrees of freedom. The NLFDM [10] transforms this continuous system into a system having a finite number of degrees of freedom. While carrying out the dynamic (or static) analysis by the NLFDM, the number of degrees of freedom depends on the number of the used nodal lines.

The natural frequencies and modes are treated in more detail because they are basic to understanding the dynamic response under any kind of excitation. It will be shown that the number of natural frequencies and that of normal modes, for axial wave number different from zero, are each equal to three times the number of nodal lines used.

The orthogonality relationships of the normal modes obtained by the nodal line finite difference method are also included. It should be remarked that these relationships are slightly different from those obtained by the finite element method; in fact the overall matrix of coefficients replaces the stiffness matrix, and any two different modes are orthogonal with respect to the unit matrix instead of the mass matrix.

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2. FORMULATION OF THE DIFFERENTIAL EQUATIONS OF MOTION

Dynamic loads are imposed on structural systems which already carry static loads (this refers to the self weight of the system). Therefore, vibrations occur about the equilibrium position that the system attains under the action of static forces.

Several approaches are commonly used to derive the governing equations of motion; e.g., Newton's second law of motion, D'lembert's principle and Hamilton's principle [12,13,16]. It should be mentioned that using D'lembert's principle is much easier than applying Newton's second law of motion to an arbitrary infinitesimal element of a thin circular cylindrical shell.

According to D'lembert's principle, dynamic equilibrium equations may be derived by extending appropriate static equilibrium equations to include inertia forces. The inertia forces acting on an arbitrary point of a circular cylindrical shell (or any thin shell of thickness h) are given as follows

$$F_I^X = -\rho h \ddot{u}$$
$$F_I^Y = -\rho h \ddot{v} \qquad (1)$$
$$F_I^Z = -\rho h \ddot{w}$$

where, F_I^X, F_I^Y and F_I^Z are the inertia forces and u, v and w are the displacements in the directions of x, y and z axes, respectively. Moreover, the double dot denotes second differentiation with respect to time, and denotes the density of the cylindrical shell material. The minus signs appear in the relations (1) because the directions of the inertia forces are opposite to those of the corresponding accelerations.

The equations of motion of a structure can be obtained from its equations of static equilibrium by adding the inertia forces acting on it to their corresponding external forces. The equations of motion of a thin circular cylindrical shell having a linear elastic isotropic material will be presented in section 4 by the use of D'lembert's principle.

3. DISPLACEMENT EQUATIONS OF MOTION

According to Agag, [10] the exact differential equations of static equilibrium of a circular cylindrical thin shell can be written as

$$[\lambda^2(u)_{20} + \frac{1-\nu}{2}(1+k)(u)_{02}] + [\frac{1+\nu}{2}\lambda(v)_{11}] + [k\lambda^3(w)_{30} - \frac{1-\nu}{2}k\lambda(w)_{12} - \nu\lambda(w)_{10}] = -\frac{R^2}{K}X \qquad (2)$$

$$[\frac{1+\nu}{2}\lambda(u)_{11}] + [\frac{1-\nu}{2}(1+3k)\lambda^2(v)_{20} + (v)_{02}] + [\frac{3-\nu}{2}k\lambda^2(w)_{21} - (w)_{01}] = -\frac{R^2}{K}Y \qquad (3)$$

$$[k\lambda^3(u)_{30} - \frac{1-\nu}{2}k\lambda(u)_{12} - \nu\lambda(u)_{10}] + [\frac{3-\nu}{2}k\lambda^2(v)_{21} - (v)_{01}] + [k\lambda^4(w)_{40} + 2k\lambda^2(w)_{22} + k(w)_{04} + 2k(w)_{02} + (1+k)w] = \frac{R^2}{K}Z \qquad (4)$$

in which the first subscript indicates the order of partial differentiation with respect to $\xi\ (\xi = \frac{x}{L})$, the second one indicates the order of partial differentiation with respect to $\frac{y}{L}$, and the number of dots indicates the order of partial differentiation with respect to time.

where

$$k = \frac{h^2}{12R^2}, \tag{5}$$

$$\lambda = \frac{R}{L} \tag{6}$$

and

$$K = \frac{Eh}{1-\nu^2} \tag{7}$$

By D'lembert's principle [16], the exact differential equations of motion of a circular cylindrical thin shell can be obtained from Eqs. (2) to (4) by adding the inertia forces, given by the relations (1) to the corresponding external forces. This procedure leads to the following partial differential equations [14, 15]

$$\left[\lambda^2(u)_{20} + \frac{1-\upsilon}{2}(1+k)(u)_{02}\right] + \left[\frac{1-\upsilon}{2}\lambda(v)_{11}\right] + \left[k\lambda^3(w)_{30} - \frac{1-\upsilon}{2}k\lambda(w)_{12} - \upsilon\lambda(w)_{10}\right] - \frac{R^2}{K}\rho h\ddot{u} = -\frac{R^2}{K}X \tag{8}$$

$$\left[\frac{1+\upsilon}{2}\lambda(u)_{11}\right] + \left[\frac{1-\upsilon}{2}(1+3k)\lambda^2(v)_{20} + (v)_{02}\right] + \left[\frac{3-\upsilon}{2}k\lambda^2(w)_{21} - (w)_{01}\right] - \frac{R^2}{K}\rho h\ddot{v} = -\frac{R^2}{K}Y \tag{9}$$

$$\left[k\lambda^3(u)_{30} - \frac{1-\upsilon}{2}k\lambda(u)_{12} - \upsilon\lambda(u)_{10}\right] + \left[\frac{3-\upsilon}{2}k\lambda^2(v)_{21} - (v)_{01}\right] + \left[k\lambda^4(w)_{40} + 2k\lambda^2(w)_{22} + k(w)_{04} + 2k(w)_{02} + (1+k)w\right] + \frac{R^2}{K}\rho h\ddot{w} = \frac{R^2}{K}Z \tag{10}$$

4. FREE VIBRATION EQUATIONS

Free vibration is the type of vibration which can occur in the absence of any externally applied forces. Free vibration occurs because of the initial conditions of displacements and velocities. If the terms representing the externally applied forces in Eqs. (8), (9) and (10) are deleted, the equations obtained are the equations of free undamped vibration for a thin circular cylindrical shell of a linear elastic isotropic material. These equations can be written as follows:

$$\left[\lambda^2(u)_{20}+\frac{1-\upsilon}{2}(1+k)(u)_{02}\right]+\left[\frac{1+\upsilon}{2}\lambda(v)_{11}\right]+$$
$$\left[k\lambda^3(w)_{30}-\frac{1-\upsilon}{2}k\lambda(w)_{12}-\upsilon\lambda(w)_{10}\right]-\frac{R^2}{K}\rho h\ddot{u}=0 \qquad (11)$$

$$\left[\frac{1+\upsilon}{2}\lambda(u)_{11}\right]+\left[\frac{1-\upsilon}{2}(1+3k)\lambda^2(v)_{20}+(v)_{02}\right]+\left[\frac{3-\upsilon}{2}k\lambda^2(w)_{21}-(w)_{01}\right]-\frac{R^2}{K}\rho h\ddot{v}=0 \qquad (12)$$

$$\left[k\lambda^3(u)_{30}-\frac{1-\upsilon}{2}k\lambda(u)_{12}-\upsilon\lambda(u)_{10}\right]+\left[\frac{3-\upsilon}{2}k\lambda^2(v)_{21}-(v)_{01}\right]+$$
$$\left[k\lambda^4(w)_{40}+2k\lambda^2(w)_{22}+k(w)_{04}+2k(w)_{02}+(1+k)w\right]+\frac{R^2}{K}\rho h\ddot{w}=0 \qquad (13)$$

5. THE NODAL LINE FINITE DIFFERENCE METHOD

The continuous interest in improving the solution techniques used for the structural analysis of two and three dimensional problems has led to the development of new semi-analytical methods. The nodal line finite difference method can be considered as one of these methods. The nodal line finite difference method (AGAG METHOD) has been established and developed by Agag [3-10]. Up to now the method has been used for linear elastic analysis of thin rectangular and circular plates, and thin circular cylindrical shells. The method treats the governing partial differential equations aiming at transforming them into ordinary differential equations, which can be solved numerically by the finite difference method.

The nodal line finite difference method (NLFDM) requires that a set of fictitious lines normal to two opposite edges of the structure should be constructed on its intermediate surface, as shown in Fig. 2, and these fictitious lines are called nodal lines. The nodal line finite difference method is based on expressing each of the displacement functions as a summation of terms, and each of these terms is the product of one term of a basic function and a nodal line parameter. It should be mentioned that the basic functions are those derived by VLASOV (1949).

6. SOLUTION OF FREE VIBRATION EQUATIONS BY THE NLFD METHOD

6.1 Basic Concept

As it has been considered in static analysis [10], the thin circular cylindrical shell has an open cross-section and it is simply supported at both the curved edges, while the longitudinal edges of the shell can be arbitrarily supported. For any natural vibration mode, each point of the shell vibrates at the same circular frequency ω, but with different amplitudes of displacements.

Using the nodal line finite difference method, the components of time dependent displacement at a nodal line numbered j can be written as

$$u_j(x,\varphi,t)=\sin(\omega t+\psi)\sum_{m=0} u_{m,j}(\varphi)\cos(m\pi\xi) \qquad (14)$$

$$v_j(x,\varphi,t)=\sin(\omega t+\psi)\sum_{m=0} v_{m,j}(\varphi)\sin(m\pi\xi) \qquad (15)$$

$$w_j(x,\varphi,t)=\sin(\omega t+\psi)\sum_{m=0} w_{m,j}(\varphi)\sin(m\pi\xi) \qquad (16)$$

where t denotes time, and ψ denotes the phase angle. It is obvious that the relations (14), (15) and (16) satisfy the boundary conditions along the two simply supported curved edges. Moreover, when they are substituted into the equations of free vibration they yield:

1. A system of three ordinary differential equations for the functions $u_{m,j}(\varphi), v_{m,j}(\varphi)$ and $w_{m,j}(\varphi)$, and this system exists for any value of the axial wave number m different from zero ($m \neq 0$). The system will be transformed into nodal line difference equations.
2. One ordinary equation for $u_{0,j}(\varphi)$ and this equation exists for only zero axial wave number. The equation has a closed form solution, which will be given latter.

The system of ordinary differential equations, obtained for m = 1 2,3, ..., can be written as

$$-d_1u''_{m,j}+d_2u_{m,j}-d_3v'_{m,j}+d_4w''_{m,j}+d_5w_{m,j}-\rho h\frac{R^2}{K}\omega^2u_{m,j}=0 \qquad (17)$$

$$d_3u'_{m,j}-v''_{m,j}+d_6v_{m,j}+d_7w'_{m,j}-\rho h\frac{R^2}{K}\omega^2v_{m,j}=0 \qquad (18)$$

$$d_4u''_{m,j}+d_5u_{m,j}-d_7v'_{m,j}+d_8w''''_{m,j}-2d_9w''_{m,j}+d_{10}w_{m,j}-\rho h\frac{R^2}{K}\omega^2w_{m,j}=0 \qquad (19)$$

in which $u_{m,j}, v_{m,j}$ and $w_{m,j}$ are functions of the variable φ, and the prime denotes ordinary differentiation with respect to this variable. Moreover, the coefficients $d_i\,(i=1,2,3,\ldots\ldots,10)$ are the same ones given by the relations

$$\left.\begin{aligned}
&d_1=\frac{1-\nu}{2}(1+k), && d_2=\lambda^2m^2\pi^2, \quad d_3=\frac{1+\nu}{2}\lambda m\pi,\\
&d_4=\frac{1-\nu}{2}k\lambda m\pi, && d_5=\nu\lambda m\pi+k\lambda^3m^3\pi^3,\\
&d_6=\frac{1-\nu}{2}(1+3k)d_2, && d_7=1+\frac{3-\nu}{2}kd_2,\\
&d_8=k, && d_9=k(d_2-1), \quad d_{10}=1+k+kd_2^2
\end{aligned}\right| \qquad (20)$$

For zero axial wave number (m=0), an ordinary differential equation is obtained and can be written as

$$d_1u''_{0,j}+\rho h\frac{R^2}{K}\omega^2u_{0,j}=0 \qquad (21)$$

6.2 Nodal Line Difference Equations

The nodal line difference equations corresponding to the system of the ordinary differential equations (17), (18) and (19) can be obtained by replacing the ordinary derivatives in these differential equations with their corresponding finite difference expressions. Using the central finite difference expression corresponding to the derivatives of the first, second and fourth order, a system of three nodal line difference equations will be obtained for each nodal line [10,11]. This system can be written in the following matrix form

$$[A]\{\delta_{m,j-2}\}+[B]\{\delta_{m,j-1}\}+[C]\{\delta_{m,j}\}+[B]^T\{\delta_{m,j+1}\}+ [A]^T\{\delta_{m,j+2}\}-\rho h\frac{R^2}{K}\Delta^2\omega^2\{\delta_{m,j}\}=\{0\} \tag{22}$$

where each of the matrices [A], [B] and [C] is a square matrix of order three, and these matrices are the same ones given by the relations

$$[A]=\begin{bmatrix}0&0&0\\0&0&0\\0&0&c_8\end{bmatrix},\ [B]=\begin{bmatrix}-c_1&c_2&c_3\\-c_2&-1&-c_6\\c_3&c_6&-c_9\end{bmatrix},\ [C]=\begin{bmatrix}c_4&0&c_5\\0&c_7&0\\c_5&0&c_{10}\end{bmatrix},$$

$$c_1=d_1,\ c_2=d_3\Delta/2,\ c_3=d_4,\ c_4=2d_1+d_2^2, \tag{23}$$

$$c_5=d_5\Delta^2-2d_4,\ c_6=d_7\Delta/2,\ c_7=2+d_6\Delta^2,\ c_8=d_8/\Delta^2,$$

$$c_9=(4d_8/\Delta^2)+2d_9,\ c_{10}=(6d_8/\Delta^2)+4d_9+d_{10}\Delta^2,\ \text{and}\ \Delta=\Delta\varphi$$

Furthermore, the vectors $\{\delta_{m,j-2}\},\{\delta_{m,j-1}\},\{\delta_{m,j}\},\{\delta_{m,j+1}\}$, and $\{\delta_{m,j+2}\}$ are defined as follows

$$\{\delta_{m,k}\}=[u_{m,k}\ v_{m,k}\ w_{m,k}]^T, k=j-2, j-1, j, j+1,\ \text{or}\ j+2 \tag{24}$$

It should be remarked that the elements of the matrices [A], [B] and [C] are functions of Poisson's ratio υ, the ratio $\frac{R}{h}$, the ratio $\frac{L}{mR}$ and $\Delta\varphi$. This can be proved by making use of the relations (20) and (23). In addition to that, both matrices [A] and [C] are symmetric.

The application of the matrix equation at each nodal line yields a system of 3N linear homogeneous algebraic equations, where N is the number of nodal lines used. Then, the homogeneous boundary conditions along the two longitudinal edges are transformed into nodal line difference equations and are used of to eliminate the parameters of the exterior nodal lines from the system. Finally, the following matrix equation can be obtained

$$[k_m]\{x_m\}=\rho h\frac{R^2}{K}\omega^2\Delta^2\{x_m\} \tag{25}$$

wher the matrix $[k_m]$ is the overall matrix of coefficients, and the vector $\{x_m\}$ is given by the following relations

$$\{x_m\}=[[\delta_{m,1}][\delta_{m,2}]\ldots\ldots\ldots\ldots[\delta_{m,N}]]^T, \tag{26}$$

$$[\delta_{m,k}]=[u_{m,k}\ v_{m,k}\ w_{m,k}],\ k=1,2,3,\ldots\ldots\ldots,N.$$

It should be mentioned that the matrix $[k_m]$ is a square matrix of order 3N, where N is the number of the nodal lines used. Furthermore, this matrix is symmetric if the boundary conditions along the two longitudinal edges are identical.

Equation (25) can be rewritten as

$$[k_m]\{x_m\} = \Delta^2 \frac{\rho R^2(1-\upsilon)}{E}\omega^2\{x_m\} \quad (27)$$

If

$$\Omega^2 = \frac{\rho R^2(1-\upsilon)}{E}\omega^2 \quad (28)$$

then, Eq. (27) can be rewritten as

$$[k_m]\{x_m\} = \Delta^2\Omega^2\{x_m\} \quad (29)$$

The parameter Ω is related to the circular frequency ω by the relation (28), and it is called the nondimentional frequency of the thin circular cylindrical shell.

6.3 Eigenvalue Problem

The matrix equation (29) represents a set of n algebraic homogeneous linear equations involving n unknowns, where n is equal to three times the number of nodal lines used. This set is equivalent to the set

$$([k_m] - \Delta^2\Omega^2[I])\{x_m\} = \{0\} \quad (30)$$

If

$$\beta = \Delta^2\Omega^2 \quad (31)$$

then, Eq. (30) can be rewritten as

$$([k_m] - \beta[I])\{x_m\} = \{0\} \quad (32)$$

where the matrix [I] is the unit diagonal matrix of order n.

Matrix equations having the same form as equation (32) are referred to as eigenvalue problems. Equation (32) can have non-zero solutions if and only if the determinant of the matrix $([k_m] - \beta[I])$ vanishes. Thus, the following equation is obtained

$$|[k_m] - \beta[I]| = 0 \quad (33)$$

The expansion of equation (33) leads to an n th order polynomial equation in β. The n roots $\beta_1, \beta_2, \ldots, \beta_n$ of this polynomial equation represent the eigenvalues of the matrix $[k_m]$. Because of the relation (31), the nondimensional frequency Ω also has n values; these values can be referred as $\Omega_1, \Omega_2, \ldots, \Omega_n$. Furthermore, the relation (28) means to that the circular frequency has n values; these values are referred to as $\omega_1, \omega_2, \ldots, \omega_n$. Consequently, solving the eigenvalue problem of the matrix $[k_m]$ results in n eigenvalues, n nondimensional frequencies and n circular frequencies.

Since the eigenvalue problem of the matrix $[k_m]$ must be solved for each axial wave number different from zero, then it is better to refer to the n circular frequencies as $\omega_{m,1}, \omega_{m,2}, \ldots, \omega_{m,n}$. The circular frequencies are generally arranged in ascending order with $\omega_{m,1}$ being the lowest or fundamental circular frequency.

Corresponding to each distinct circular frequency $\omega_{m,j}$ (or eigenvalue β_j), there will be a set of amplitudes or eigenvectors $\{X_m^{(j)}\}$. These amplitudes are obtained by substituting β_j for β in the matrix equation (32). Since the matrix equation (32) represents a system of algebric homogeneous linear equations, then only relative values or ratios of the amplitudes may be found. One of the amplitudes may be taken equal to unity, and then the remaining values are determined.

The eigenvector $\left\{X_m^{(j)}\right\}$ is referred to as the j th normal mode of vibration. It is obvious that the number of the normal modes, as well as the number of the circular frequencies equal to three times the number of the nodal lines used.

6.4 Orthogonality of Normal Modes

The two longitudinal edges of a thin circular cylindrical shell having simply supported curved edges are assumed to have the same boundary conditions. Therefore, the matrix $\left[k_m\right]$ is a symmetric, and the eigenvalue problem related to this matrix can be solved by Jacobi's method.

The eigenvectors or normal modes obtained from the free vibration problem satisfy certain relationships known as the orthogonality conditions. These conditions greatly facilitate computing the response of the thin circular cylindrical shell when undergoes a forced vibration motion. A derivation of the orthogonality relations follows.

A solution to the matrix equation (32) was found to be

$$\left\{X_m\right\}=\left\{X_m^{(j)}\right\} \tag{34}$$

Equation (34) represents the relative values of the amplitudes of the j th normal mode. Substituting this equation into equation (32) gives

$$\beta_j\left\{X_m^{(j)}\right\}=\left[k_m\right]\left\{X_m^{(j)}\right\} \tag{35}$$

Similarly, for free vibration motion in the i th mode,

$$\beta_i\left\{X_m^{(i)}\right\}=\left[k_m\right]\left\{X_m^{(i)}\right\} \tag{36}$$

Premultiplying Eq. (35) by the transpose of $\left\{X_m^{(i)}\right\}$, the following equation is obtained

$$\beta_j\left\{X_m^{(i)}\right\}^T\left\{X_m^{(j)}\right\}=\left\{X_m^{(i)}\right\}^T\left[k_m\right]\left\{X_m^{(j)}\right\} \tag{37}$$

Premultiplying equation (36) by the transpose of $\left[\bar{x}_m\right]$ yields

$$\beta_i\left\{X_m^{(j)}\right\}^T\left\{X_m^{(i)}\right\}=\left\{X_m^{(j)}\right\}^T\left[k_m\right]\left\{X_m^{(i)}\right\} \tag{38}$$

Making use of the symmetry of the matrix $\left[k_m\right]$ and the reversal law of transposition, the transpose of the equation (38) is

$$\left[x_m\right]^T\left[k_m\right]\left[x_m\right]=\left[\bar{k}_m\right] \tag{39}$$

Subtracting equation (39) from equation (37) yields

$$\left(\beta_j-\beta_i\right)\left\{X_m^{(i)}\right\}^T\left\{X_m^{(j)}\right\}=0 \tag{40}$$

Hence for distinct eigenvalues $\beta_i \neq \beta_j$, the following relation must be satisfied

$$\left\{X_m^{(i)}\right\}^T\left\{X_m^{(j)}\right\}=0 \tag{41}$$

Substituting the equation (41) into equation (37) yields

$$\left\{X_m^{(i)}\right\}^T\left[k_m\right]\left\{X_m^{(j)}\right\}=0 \tag{42}$$

Eqs. (41) and (42) represent the orthogonality relations which the normal modes satisfy. Equation (42) indicates that the normal modes $\{X_m^{(i)}\}$ and $\{X_m^{(j)}\}$ are orthogonal with respect to the matrix $[k_m]$, but equation (41) indicates that they are orthogonal with respect to the unit diagonal matrix of order n.

If i and j refer to the same mode, so that $\beta_i = \beta_j$, then the left hand side of Eq. (41) is, in general, a nonzero constant. This constant is denoted by M_j; that is

$$\{X_m^{(j)}\}^T \{X_m^{(j)}\} = 0 \qquad (43)$$

Substitution of Eq. (43) into Eq. (37) gives, for the case of i=j,

$$\{X_m^{(j)}\}^T [k_m] \{X_m^{(j)}\} = \beta_j M_j \qquad (44)$$

If $\{X_m^{(j)}\}$ represents the jth column of the so-called "modal matrix" $[X_m]$, the orthogonality conditions may also be written as

$$[x_m]^T [x_m] = [\bar{x}_m]$$
$$[x_m]^T [k_m][x_m] = [\bar{k}_m] \qquad (45)$$

where the matrices $[\bar{x}_m]$ and $[\bar{k}_m]$ are diagonal ones and the elements of which are defined as

$$\bar{x}_m(i,j) = M_j$$
$$\bar{k}_m(i,j) = \beta_j M_j \ , \ j = 1,2,3,\ldots\ldots\ldots\ldots,n \qquad (46)$$

6.5 General Solution of Free Vibration Equations

Because the normal modes have the property of orthogonality, they represent n independent solutions, where n is equal to three times the number of the nodal lines used. Therefore, the general solution to the problem of free vibration is a linear combination of all these modes. The general solution can be written as follow

$$U_j(x,\varphi,t) = \sum_{m=1} \Big[\sum_{r=1} U_{m,j}^{(r)} C_{m,r} \sin(\omega_{m,r} t + \psi_{m,r}) \Big] \cos(m\pi\xi) + \underline{U_{m=0}(\varphi_j, t)}$$
$$V_j(x,\varphi,t) = \sum_{m=1} \Big[\sum_{r=1} V_{m,j}^{(r)} C_{m,r} \sin(\omega_{m,r} t + \psi_{m,r}) \Big] \sin(m\pi\xi)$$
$$W_j(x,\varphi,t) = \sum_{m=1} \Big[\sum_{r=1} W_{m,j}^{(r)} C_{m,r} \sin(\omega_{m,r} t + \psi_{m,r}) \Big] \sin(m\pi\xi) \qquad (47)$$

where $U_{m,j}^{(r)}, V_{m,j}^{(r)}$ and $W_{m,j}^{(r)}$ denote the relative amplitudes of the axial, tangential and radial displacements, respectively, at the jth nodal line when the r th mode exist. Moreover, the underlined term is the function represseenting the axial displacement corresponding to zero axial wave number. The explicit form of this function will be given in section 7.6.

The relation (47) contains 2n arbitrary constants $C_{m,1}, C_{m,2}, \ldots, C_{m,n}$, $\psi_{m,1}, \psi_{m,2}, \ldots, \psi_{m,n}$, for even terms of series, which can be completely determined from the prescribed initial condition. Therefore, the number of the initial conditions must be equal to 2n (six time the number of the nodal line used).

Expanding the initial displacements and velocities in Fourier series with the fundamental period 2L, the following relations are obtained

$$Uo_j = \sum_{m=o} Uo_{m,j}\cos(m\pi\xi)\ , \qquad \dot{U}o_j = \sum_{m=o} \dot{U}o_{m,j}\cos(m\pi\xi) \tag{48a}$$

$$Vo_j = \sum_{m=o} Vo_{m,j}\sin(m\pi\xi)\ , \qquad \dot{V}o_j = \sum_{m=o} \dot{V}o_{m,j}\sin(m\pi\xi) \tag{48b}$$

$$Wo_j = \sum_{m=o} Wo_{m,j}\sin(m\pi\xi)\ , \qquad \dot{W}o_j = \sum_{m=o} \dot{W}o_{m,j}\sin(m\pi\xi) \tag{48c}$$

where Uo_j, Vo_j and Wo_j denote the initial displacements of the nodal line numbered j, and $\dot{U}o_j, \dot{V}o_j$ and $\dot{W}o_j$ denotes the initial velocities of the same nodal line.

Applying the relations (47) after putting t equal to zero at each nodal line and making use of the relations (48), the following matrix equations are obtained for each axial wave number different from zero

$$\{\delta o_m\} = [x_m]\{A\} \tag{49}$$

$$\{\dot{\delta} o_m\} = [x_m]\{\omega B\} \tag{50}$$

where

$$\begin{aligned}
\{\delta o_m\} &= [Uo_{m,1}\ Vo_{m,1}\ Wo_{m,1} \ldots\ldots Uo_{m,N}\ Vo_{m,N}\ Wo_{m,N}]^T,\\
\{\dot{\delta} o_m\} &= [\dot{U}o_{m,1}\ \dot{V}o_{m,1}\ \dot{W}o_{m,1} \ldots\ldots \dot{U}o_{m,N}\ \dot{V}o_{m,N}\ \dot{W}o_{m,N}]^T,\\
\{A\} &= [A_{m,1}\ A_{m,2} \ldots\ldots A_{m,N}]^T,\\
\{\omega B\} &= [\omega_{m,1}B_{m,1}\ \omega_{m,2}B_{m,2} \ldots\ldots \omega_{m,N}B_{m,N}]^T,\\
A_{m,r} &= C_{m,r}\sin(\psi_{m,r}), B_{m,r} = C_{m,r}\cos(\psi_{m,r}),\ r = 1,2,\ldots\ldots,n,
\end{aligned} \tag{51}$$

where **N** is the number of the nodal line used, n = 3N.

Solving the matrix equation (49) and (50) for the arbitrary constants gives

$$\{A\} = [x_m]^{-1}\{\delta o_m\} \tag{52}$$

$$\{\omega B\} = [x_m]^{-1}\{\dot{\delta} o_m\} \tag{53}$$

Taking into consideration the relations

$$A_{m,r} = C_{m,r}\sin(\psi_{m,r}), B_{m,r} = C_{m,r}\cos(\psi_{m,r}),\ r = 1,2,\ldots\ldots,n \tag{54}$$

the general solution can be rewritten as follows

$$\begin{aligned}
U_j(x,\varphi,t) &= \sum_{m=1}\Big[\sum_{r=1}^{n}\{A_{m,r}\cos(\omega_{m,r}t) + B_{m,r}\sin(\omega_{m,r}t)\}U_{m,j}^{(r)}\Big]\cos(m\pi\xi)\\
&\qquad + U_{m=o}(\varphi_j,t)\\
V_j(x,\varphi,t) &= \sum_{m=1}\Big[\sum_{r=1}^{n}\{A_{m,r}\cos(\omega_{m,r}t) + B_{m,r}\sin(\omega_{m,r}t)\}V_{m,j}^{(r)}\Big]\sin(m\pi\xi)\\
W_j(x,\varphi,t) &= \sum_{m=1}\Big[\sum_{r=1}^{n}\{A_{m,r}\cos(\omega_{m,r}t) + B_{m,r}\sin(\omega_{m,r}t)\}W_{m,j}^{(r)}\Big]\sin(m\pi\xi)
\end{aligned} \tag{55}$$

The rth mode, for example, represents the relative values of the parameters of the nodal lines when this mode exists; therefore, it is given by the relation

$$\{x_m^{(r)}\} = [U_{m,1}^{(r)}\ V_{m,1}^{(r)}\ W_{m,1}^{(r)} \ldots\ldots U_{m,N}^{(r)}\ V_{m,N}^{(r)}\ W_{m,N}^{(r)}]^T \tag{56}$$

N= the number of nodal lines used.

Once the natural frequencies and the normal modes are determined, the values of the arbitrary constants $A_{m,1}, A_{m,2}, \ldots\ldots, A_{m,n}$, $B_{m,1}, B_{m,2}, \ldots\ldots, B_{m,n}$ can be completely obtained from the relations (52) and (53). Then, substituting these values into the relations (55), the values of the displacements at any point belonging to each nodal line can be obtained. It should be mentioned that the velocities in the directions of x, j and z axes, at any point belonging to each nodal line, can be determined from the relations (55) by differentiating these relations once with respect to time. Moreover, the accelerations can also be obtained by differentiating the same relations twice with respect to time.

6.6 Natural Modes for Zero Axial Wave Number

For zero axial wave number (m=0), the natural modes are purely longitudinal ($W_{o,j} = V_{o,j} = 0$ and $U_{o,j} \neq 0$). These modes can be obtained by solving the homogenous ordinary differential equation (21). The closed form solution of this equation will be derived below.

The ordinary differential equation (21) can be rewritten as follows

$$d_1 \underset{m=o}{U''(\varphi)} + \Omega^2 \underset{m=o}{U(\varphi)} = 0 \tag{57}$$

where

$$d_1 = \frac{1-\upsilon}{2}, \qquad \Omega^2 = (\frac{1-\upsilon^2}{E})\rho R^2 \omega^2 \tag{58}$$

Dividing each term of the above ordinary differential equation by d_1, the following ordinary differential equation is obtained

$$\underset{m=o}{U''(\varphi)} + f^2 \underset{m=o}{U(\varphi)} = 0 \tag{59}$$

where

$$f^2 = \frac{\Omega^2}{d_1} \tag{60}$$

Eq. (59) is an ordinary differential equation of the second order, which has the following general solution

$$\underset{m=o}{U(\varphi)} = C_1 \cos(f\varphi) + C_2 \sin(f\varphi) \tag{61}$$

where C_1 and C_2 are two arbitrary constants which can be determined from the boundary conditions.

One of the following 2 sets of boundary conditions must be satisfied along each longitudinal edge of the circular cylindrical thin shell

$$\underset{m=o}{U(\varphi)} = 0; \qquad \underset{m=o}{N_{\varphi x}(\varphi)} = 0 \ \ (\text{or } \underset{m=o}{U'(\varphi)} = 0) \tag{62}$$

Assuming that each of the two longitudinal edges has a zero axial displacement boundary conditions and letting the angle φ be measured as shown in figure (3), the following relations can be obtained

$$C_1 = 0 \ \text{ and } \ C_2 \sin(f\varphi o) = 0 \tag{63}$$

Since C_2 must not be equal to zero (in order to obtain a nontrivial solution), then

$$\sin(f\varphi o) = 0 \tag{64}$$

Upon solving the trigonometric equation (64), the following relation is obtained

$$f = \frac{r\pi}{\varphi o}, \; r = 1,2,3,\ldots\ldots \tag{65}$$

The relation (65) indicates that f has an infinite number of values. Therefore, it is better to write this relation as

$$f_r = \frac{r\pi}{\varphi o}, \; r = 1,2,3,\ldots\ldots \tag{66}$$

Making use of the relation (60), the nondimensional frequencies corresponding to zero axial wave number can be given by the following relation

$$\Omega_{o,r} = \frac{r\pi}{\varphi o}\sqrt{\frac{1-\upsilon}{2}}, \; r = 1,2,3,\ldots\ldots \tag{67}$$

where $\Omega_{o,r}$ denotes the nondimensional frequency of the r th mode for zero axial wave number (m=0).

The relation (66) gives the eigenvalues of the homogeneous ordinary differential equation (59), and the corresponding eigenfunctions can be written as

$$\underset{m=o}{\overset{(r)}{U}}(\varphi) = \sin\left(\frac{r\pi\varphi}{\varphi o}\right), \; r = 1,2,3,\ldots\ldots \tag{68}$$

The eigenfunctions, which are given by the relation (68), represent the axial modes for zero axial wave number (m=0). It should be noted that these modes are the only ones corresponding to zero axial wave number. furthermore, these modes satisfy the following relation

$$\int_0^{\varphi o} \underset{m=o}{U^{(r)}}\,\underset{m=o}{U^{(s)}}\, d\varphi = 0, \; r \neq s \tag{69}$$

The relation (69), which can be obtained by direct integration, indicates that any two different axial modes corresponding to zero axial wave number are orthogonal to each other.

Since the eigenfunctions (or the axial normal modes), which are given by the relation (68), are orthogonal to each other, they represent an infinite number of independent solutions to the ordinary differential equation (59). Therefore, the general solution to this ordinary differential equation is a linear combination of these eigenfunctions (or the axial normal modes). Making use of the relations (14) and (68), the axial displacement part corresponding to zeroaxial wave number can be written as

$$\underset{m=o}{U}(x,\varphi,t) = \underset{m=o}{U}(,\varphi,t) = \sum_{r=1} C_{o,r} \sin(\omega_{o,r} t + \psi_{o,r}) \sin\left(\frac{r\pi\varphi}{\varphi o}\right) \tag{70}$$

The relation (70) can be rewritten in the following form

$$\underset{m=o}{U}(,\varphi,t) = \sum_{r=1} \left\{A_{o,r}\cos(\omega_{o,r} t) + B_{o,r}\sin(\omega_{o,r} t)\right\} \sin\left(\frac{r\pi\varphi}{\varphi o}\right) \tag{71}$$

Differentiating each side of the above equations with respect to the variable t, the following relation is obtained

$$\underset{m=o}{\dot{U}}(,\varphi,t) = \sum_{r=1} \omega_{o,r} \left\{-A_{o,r}\sin(\omega_{o,r} t) + B_{o,r}\cos(\omega_{o,r} t)\right\} \sin\left(\frac{r\pi\varphi}{\varphi o}\right) \tag{72}$$

The relation (72) represents the axial velocity part corresponding to zero axial wave number (m=0).

Expanding the initial axial displacement as well as the initial axial velocity in Fourier series with the fundamental period 2L, the following relations are obtained

$$Uo(x,\varphi) = \sum_{m=o} \underset{m}{Uo}(\varphi)\cos(m\pi\xi), \quad \dot{U}o(x,\varphi) = \sum_{m=o} \underset{m}{\dot{U}o}(\varphi)\cos(m\pi\xi), \;\; \xi = \frac{x}{L} \qquad (73)$$

Upon substitution for $t = 0$ in Eqs. (71) and (72) and making use of Eq. (73), the following relations can be obtained

$$\underset{m=o}{Uo}(\varphi) = \sum_{r=1} A_{o,r} \sin(\frac{r\pi\varphi}{\varphi o}), \;\; \underset{m=o}{\dot{U}o}(\varphi) = \sum_{r=1} \omega_{o,r} B_{o,r} \sin(\frac{r\pi\varphi}{\varphi o}) \qquad (74)$$

It should be observed that the left hand sides of the relations (74) represent the initial axial displacement part and the initial axial velocity part, respectively, corresponding to zero axial wave number.

Formulas (74) are expansions of the Fourier series in sines ii. the interval $[0, \varphi o]$; therefore, the coefficients $A_{o,r}$ and $B_{o,r}$ can be obtained from the following relations

$$A_{o,r} = \frac{2}{\varphi o} \int_o^{\varphi o} \underset{m=o}{Uo}(\varphi) \sin(\frac{r\pi\varphi}{\varphi o}) d\varphi, \;\; r = 1,2,3,........ \qquad (75)$$

$$B_{o,r} = \frac{2}{\omega_{o,r}\varphi o} \int_o^{\varphi o} \underset{m=o}{Uo}(\varphi) \sin(\frac{r\pi\varphi}{\varphi o}) d\varphi, \;\; r = 1,2,3,........ \qquad (76)$$

Upon the determination of the coefficients $A_{o,r}$ and $B_{o,r}$ (r=1,2,...), the axial displacement part as well as the axial velocity part corresponding to zero axial wave number can be determined at any point of the circular cylindrical thin shell.

It should be observed that the axial displacement part and the axial velocity part corresponding to zero axial wave number are each independent of the axial coordinate (x-coordinate). Moreover, for free vibration motion, if both the initial axial displacement and velocity at each point of the circular cylindrical shell are equal to zero, the axial displacement part and the axial velocity part corresponding to zero axial wave number do not exist. In such cases, there is no need to determine the axial modes corresponding to zero axial wave number.

7. BOUNDARY CONDITIONS

The boundary conditions along the two longitudinal edges of a thin circular cylindrical shell, may be static, kinematic or mixed, depending on the method of supporting these edges. The boundary conditions along a free edge are static, and the boundary conditions along a rigidly fixed edge are kinematic.

Consider a thin circular cylindrical shell with an open cross section and simply supported curved edges. If this circular cylindrical shell has identical boundary conditions along its two longitudinal edges and is symmetrically or antisymmetrically loaded, then half of it is considered and only four boundary conditions along the longitudinal edge of the considered half are needed in structural analysis. This property has been used to reduce the number of the nodal lines used while carrying out the static analysis of a thin circular cylindrical shell by the nodal line finite difference method [10].

For a thin circular cylindrical shell of an open cross section, it should be remarked that if each two opposite edges have the same boundary conditions, then the natural modes of vibration are either symmetric or antisymmetric. In such cases, half of the circular cylindrical shell is considered, and each type of the natural modes is determined alone. This property is utilized to reduce the number of the nodal lines used while determining the natural modes and the corresponding frequencies by the nodal line finite difference method.

There are 16 sets of boundary conditions, one of which must be satisfied along each longitudinal edge of a thin circular cylindrical shell with an open cross section [14,15]. Some well known sets of boundary conditions will be described below. One of these sets may exist along a longitudinal edge:

1. Free edge

$$N_\varphi = 0,\ M_\varphi = 0,\ N_{\varphi x} = 0,\ \Omega_\varphi + (M_{\varphi x})_{10} = 0 \tag{77}$$

where the first and second numerical subscripts indicates the order of partial differentiation with respect to the variables x and φ respectively.

2. Clambed edge (rigidly fixed edge)

$$u = 0,\ v = 0,\ w = 0,\ (w)_{01} = 0 \tag{78}$$

3. Edge with hinged immovable support

$$u = 0,\ v = 0,\ w = 0,\ M_\varphi = 0 \tag{79}$$

4. Edge with hinged movable support in the normal direction

$$u = 0,\ v = 0,\ \Omega_\varphi + (M_{\varphi x})_{10},\ M_\varphi = 0 \tag{80}$$

5. Edge with hinged movable support in the direction of φ -axis

$$u = 0,\ N_\varphi = 0,\ w = 0,\ M_\varphi = 0 \tag{81}$$

6. Edge with hinged movable support in the tangent plane

$$N_{\varphi x} = 0,\ N_\varphi = 0,\ w = 0,\ M_\varphi = 0 \tag{82}$$

7. Edge with hinged movable support in the direction of x-axis and the normal plane

$$N_{\varphi x} = 0,\ v = 0,\ \Omega_\varphi + (M_{\varphi x})_{10},\ M_\varphi = 0 \tag{83}$$

It should be mentioned that the displacement functions given by the relations (14), (15) and (16) satisfy the boundary conditions along the simply supported curved edges. For a thin circular cylindrical shell with an open cross-section and simply supported curved edges, the following stress resultant displacement relations hold.

$$\begin{aligned}
N_\varphi &= \frac{K}{R}\sin(\omega t + \psi) \sum_{m=1} N_\varphi^m(\varphi)\sin(m\pi\xi) \\
N_{\varphi x} &= \frac{K}{R}\left(\frac{1-\upsilon}{2}\right)\sin(\omega t + \psi) \sum_{m=0} N_{\varphi x}^m(\varphi)\cos(m\pi\xi) \\
M_\varphi &= -\frac{D}{R^2}\sin(\omega t + \psi) \sum_{m=1} M_\varphi^m(\varphi)\sin(m\pi\xi) \\
\Omega_\varphi + (M_{\varphi x})_{10} &= -\frac{D}{R^3}\sin(\omega t + \psi) \sum_{m=1} [\Omega_\varphi^m(\varphi)]_{eff} \sin(m\pi\xi)
\end{aligned} \tag{84}$$

where

$$\begin{aligned}
N_\varphi^m(\varphi) &= -\upsilon\lambda_m u_m + v'_m - w_m - k(w_m + w''_m) \\
N_{\varphi x}^m(\varphi) &= u'_m + \lambda_m v_m + k(u'_m - \lambda_m w''_m) \\
M_\varphi^m(\varphi) &= w''_m + (1-\upsilon\lambda_m^2)w_m \\
[\Omega_\varphi^m(\varphi)]_{eff} &= (\frac{1-\upsilon}{2})\lambda_m u'_m - \frac{3(1-\upsilon)}{2}\lambda_m^2 v_m + (1-2\lambda_m^2+\upsilon\lambda_m^2)w'_m + w'''_m
\end{aligned} \tag{85}$$

where the prime denotes ordinary differentiation with respect to the variable φ, and $\lambda_m = \lambda m\pi$.

Solving the free vibration problem of a circular cylindrical thin shell by the nodal line finite difference method requires expanding the initial displacements and velocities into Fourier series. The natural frequencies and their corresponding modes are determined for each term of these expansions or for each value of the axial wave number m. Therefore, the boundary conditions along each longitudinal edge of the circular cylindrical shell must be formulated for each value of the axial wave number.

For each axial wave number different from zero, some well known cases of boundary conditions will be given below. One of these cases may exist along a longitudinal edge

1. Free edge

$$N_\varphi^m(\varphi) = 0,\ M_\varphi^m(\varphi) = 0,\ N_{\varphi x}^m(\varphi) = 0,\ [\Omega_\varphi^m(\varphi)]_{eff} = 0 \tag{86}$$

2. Clamped edge (rigidly fixed edge)

$$u_m = 0,\ v_m = 0,\ w_m = 0,\ w'_m = 0 \tag{87}$$

3. Edge with hinged immovable support

$$u_m = 0,\ v_m = 0,\ w_m = 0,\ M_\varphi^m(\varphi) = 0 \tag{88}$$

4. Edge with hinged movable support in the normal direction

$$u_m = 0,\ v_m = 0,\ [\Omega_\varphi^m(\varphi)]_{eff} = 0,\ M_\varphi^m(\varphi) = 0 \tag{89}$$

5. Edge with hinged movable support in the direction of φ-axis

$$u_m = 0,\ N_\varphi^m(\varphi) = 0,\ w_m = 0,\ M_\varphi^m(\varphi) = 0 \tag{90}$$

6. Edge with hinged movable support in the tangent plane

$$N_{\varphi x}^m(\varphi) = 0,\ N_\varphi^m(\varphi) = 0,\ w_m = 0,\ M_\varphi^m(\varphi) = 0 \tag{91}$$

7. Edge with hinged movable support in the direction of x-axis and the normal plane

$$N_\varphi^m(\varphi) = 0,\ v_m = 0,\ [\Omega_\varphi^m(\varphi)]_{eff} = 0,\ M_\varphi^m(\varphi) = 0 \tag{92}$$

For zero axial wave number, one of the following two sets of boundary conditions must be satisfied along each longitudinal edge of the circular cylindrical shell

$$\underset{m=0}{u}(\varphi) = 0;\ \overset{m=0}{N_{\varphi x}}(\varphi) = 0\ (\text{or}\ \underset{m=0}{u'}(\varphi) = 0) \tag{93}$$

Making use of the relations (85), the boundary conditions along each longitudinal edge can be expressed in terms of the parameters of the edge nodal line ($\underset{m,edge}{u(\varphi)}$, $\underset{m,edge}{v(\varphi)}$ and $\underset{m,edge}{w(\varphi)}$) and their derivatives. The derivatives are not suitable for the nodal line finite difference method, and they must be converted into nodal line difference expressions. Then, the boundary conditions along the left longitudinal edge are used for eliminating the parameters of the left exterior nodal lines. The same procedure must be carried out for the boundary conditions along the right longitudinal edge.

The parameters of the exterior nodal lines can easily be eliminated from the nodal line difference equations of free vibration motion. For some well known cases of boundary conditions, Agag [10] has expressed the parameters of the exterior nodal lines in terms of those of the corresponding edge and those of its two adjacent interior nodal lines.

8. CONCLUSION

In this paper, the nodal line finite difference method has been extended to the analysis of a thin circular cylindrical shell of linear elastic isotropic material, undergoing free vibrations. The method has been applied and verified with many examples in a complementary paper [2] published in the same periodical.

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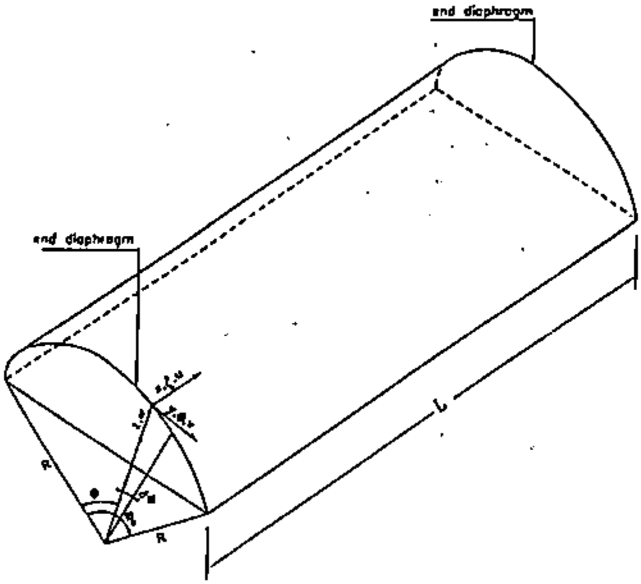
10. LIST OF SYMPOLS

The following sympols are used in this paper

F_I^X, F_I^Y, F_I^Z	= inertia forces in direction of x, y and z respectively	E	= modulus of elasticity
ρ	= material density	h	= thickness of shell
u, v, w	= displacements in direction of x, y and z respectively	R	= radius of shell
$\ddot{u}, \ddot{v}, \ddot{w}$	= accelerations in direction of x, y and z respectively	L	= length of shell
ν	= Poisson's ratio	ω	= natural circular frequency
		ψ	= phase angle
		m	= number of axial waves
		Ω	= nondimensional frequency parameter


end diaphragm
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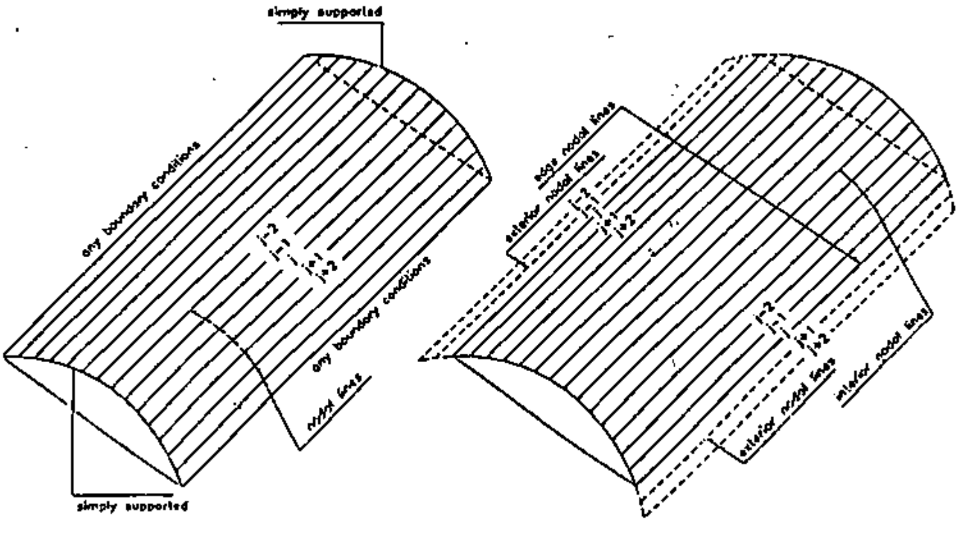
R
R
L

Fig. 1.


simply supported
any boundary conditions
any boundary conditions
nodal lines
simply supported
edge nodal lines
exterior nodal lines
exterior nodal lines
interior nodal lines

(a) (b)

Fig. 2.