Mansoura Engineering Journal

Volume 47 | Issue 6 Article 13

12-1-2022

Numerical Solution of Biharmonic Equation Using Modified Bi-Quintic B-Spline Collocation Method

Atallah El-shenawy
Mathematics and Physics Department, Faculty of Engineering, Mansoura University, Mansoura, Egypt -Assistant Professor at Department of Mathematics, Faculty of Science, New Mansoura University, New Mansoura, Egypt, atallahtm@yahoo.com

Mohamed El-Gamel
Mathematics and Physics Department, Faculty of Engineering, Mansoura University, Mansoura, Egypt, gameleg@yahoo.com

Dina Reda Jaheen
Department of basic of sciences, Mansoura Higher Institute for Engineering and Technology, Mansoura, Egypt, dmatar@mc.edu.eg

Recommended Citation

El-shenawy, Atallah; El-Gamel, Mohamed; and Jaheen, Dina Reda (2022) "Numerical Solution of Biharmonic Equation Using Modified Bi-Quintic B-Spline Collocation Method," *Mansoura Engineering Journal*: Vol. 47 : Iss. 6 , Article 13.
Available at: https://doi.org/10.58491/2735-4202.3173

Mansoura University
Faculty of Engineering
Mansoura Engineering Journal

Numerical Solution of Biharmonic Equation Using Modified Bi-Quintic B-Spline Collocation Method

Atallah El-shenawy, Mohamed El-Gamel and Dina Reda*

KEYWORDS:
Modified Quintic B-spline, Biharmonic equations, Dirichlet problem, Collocation method

Abstract*—In the following work, the numerical investigation of biharmonic equation is explored. The approximate solution is approximated at specific points in the solution domain by using the collocation method based on the modified bi-quintic b-spline basis functions. These modified basis functions vanish at the boundary points. The main properties of these basis functions are discussed in detail. The method is based on reducing the proposed problem to a linear system of equations. The boundary conditions are combined in the resulting linear system of equations in specific order to guarantee that the approximate solution coincides with the exact solution at the boundary points. Three numerical examples show the effectiveness of our method, and the accuracy is measured by comparing three different types of error between approximate and exact solutions. The outcomes are graphically depicted to assess the performance of the intended method. The proposed method is easy to implement, and numerical results ensure that the method approximates the solution of the biharmonic problem very well.

I.INTRODUCTION

AMONG the most significant problems in the field of physics and engineering applications is the biharmonic equation. For example, the deformation of a thin plate, the modeling of bio-fluid dynamics and motion of a fluid, etc. All of these applications are modeled by the biharmonic equation. Also, an important role is played by the biharmonic equations in the applications of quantum mechanics, gravitational theory and structural and continuum mechanics. The biharmonic problem was solved analytically to obtain a closed-form solution by using the separation of variables method. Recently it was solved by various analytical methods such as variational iteration, decomposition, Homotopy and the Fourier-Yang integral transform methods [1–4]. The numerical methods used for solving such types of problems have been widely discussed and can be classified into major classes. The spectral methods, finite element method and finite difference method.

The method of finite difference uses a technique that discretized the solution domain into grid points with an equal

Received: (26 July, 2022) - Revised: (04 October, 2022) - Accepted: (24 October, 2022)

Atallah El-shenawy, Assistant Professor at Mathematics and Physics Department, Faculty of Engineering, Mansoura University, Mansoura, Egypt - Assistant Professor at Department of Mathematics, Faculty of Science, New Mansoura University, New Mansoura, Egypt (e-mail: atallahtm@yahoo.com).

Mohamed El-Gamel, Professor at Mathematics and Physics Department, Faculty of Engineering, Mansoura University, Mansoura, Egypt (e-mail: gameleg@yahoo.com).

****Corresponding Author**: Dina Reda, Instructor at Department of basic of sciences, Mansoura Higher Institute for Engineering and Technology, Mansoura, Egypt (e-mail: dmatar@mc.edu.eg).*

step size. Such methods were presented by many authors as [5–7]. The main advantage of finite element method is it can be implemented in flexible geometry and can be found in detail [8–12]. Recently, spectral methods are commonly used to solve such types of problems by approximating the solution as a series of basis functions. Here we can cite some of the most famous spectral techniques for example, Legendre [13], Chebyshev tau meshless method [14], Haar wavelets implemented in [15], and Trefftz method in [16]. Sinc-Galerkin method was introduced in [17]. The localized radial basis functions collocation method was used in [18].

Spectral methods such as collocation method is widely implemented to find the approximate solution to both ordinary and partial differential equations appearing in physical phenomena as well as engineering models. Due to the very important properties of b-spline basis, they have been induced with such methods for approximating the solution of various problems. Chang et al. [19] used cubic B-spline basis functions to solve numerically the linear ordinary differential equations. Caglar et al. [20] introduced the same technique to find the numerical solution of linear system of 2nd order boundary value problems. The method based on using cubic b-spline collocation was extended to solve higher ODE by Khalid et al. in [21] where the sixth order boundary value problems were solved. Another paper proposed the solution of linear fourth order BVP by Gupta and Kumar [22]. Further approaches deal with higher degree b-splines for approximating the solution of some well-known applied problems have been discussed in detail in [23–25].

On the other side, many authors proposed novel techniques to approximate the solution of partial differential equations such as singularly perturbed convection diffusion problem [26], the solution of Burger's equation using B-spline finite element algorithms [27]. Mittal and Arora proposed an approach based on quintic B-spline collocation for solving the Kuramoto-Sivashinsky equation in [28]. Various techniques were examined by using different degrees of b-spline bases for solving heat equation, the equal width (EW) equation, the nonlinear Korteweg-de-Vries Burgers equation and Euler Bernoulli beam models in [29–32], respectively.

The biharmonic problem has consisted of a fourth-order linear differential equation with boundary conditions of type Dirichlet and Neumann. It can be described as follows:

Assume the non-homogeneous biharmonic equation in the form

$$\Lambda\,[\psi] = \eta(x,y), (x,y) \in \Omega, \tag{1.1}$$

Where the differential operator Λ is defined by

$$\Lambda[\psi] = \nabla^4\psi(x,y) = \frac{\partial^4\psi(x,y)}{\partial x^4} + \frac{\partial^4\psi(x,y)}{\partial x^2 \partial y^2} + \frac{\partial^4\psi(x,y)}{\partial y^4},$$

With respect to the following boundary conditions

$$\psi|_{\partial\Omega} = f, \tag{1.2}$$

$$\left.\frac{\partial\psi}{\partial n}\right|_{\partial\Omega} = g. \tag{1.3}$$

Where $\Omega \subset \mathbb{R}^2$ is a simply connected domain and $\partial\Omega$ its piecewise (smooth) boundaries. The boundary conditions consist of two equations, the first is Dirichlet type and describes the values of the unknown function on $\partial\Omega$, and the other condition of Neumann type where $\frac{\partial\psi}{\partial n}$ is the outward normal derivative of ψ of the boundaries $\partial\psi$.

II. SCHEME OF SOLUTION

2.1. Basic Formulas of Quintic B-Spline Basis Functions.

Assume the uniform 1D grid points $\zeta_i, i = 0,1,\cdots,N$ in the interval $[a,b]$ which are divided uniformly with

$$h = \zeta_{k+1} - \zeta_k = \frac{(b-a)}{N}$$

$$\pi : a = \zeta_0 < \zeta_1 < \cdots < \zeta_N = b$$

TABLE 1.
THE VALUES OF $\phi_i(\zeta)$ AND THE FIRST FOUR DERIVATIVES $\phi_i'(\zeta)$, $\phi_i''(\zeta)$, $\phi_i'''(\zeta)$, $\phi_i^{(iv)}(\zeta)$ AT THE NODAL POINTS.

ζ	ζ_{i-3}	ζ_{i-2}	ζ_{i-1}	ζ_i	ζ_{i+1}	ζ_{i+2}	ζ_{i+3}
$\phi_i(\zeta)$	0	1	26	66	26	1	0
$h\phi_i'(\zeta)$	0	5	50	0	−50	−5	0
$h^2\phi_i''(\zeta)$	0	20	40	−120	40	20	0
$h^3\phi_i'''(\zeta)$	0	60	−120	0	120	−60	0
$h^4\phi_i^{(iv)}(\zeta)$	0	120	−480	720	−480	120	0

The quintic B-splines are continuously differentiable, piecewise fifth degree polynomials defined on the interval $[\mathrm{a},\mathrm{b}]$ which are in the form [33]:

$$\phi_k(\zeta) = \frac{1}{h^5}\begin{cases} (\zeta-\zeta_{k-3})^5, & \zeta_{k-3} \le \zeta < \zeta_{k-2} \\ (\zeta-\zeta_{k-3})^5 - 6(\zeta-\zeta_{k-2})^5, & \zeta_{k-2} \le \zeta < \zeta_{k-1} \\ (\zeta-\zeta_{k-3})^5 - 6(\zeta-\zeta_{k-2})^5 + 15(\zeta-\zeta_{k-1})^5, & \zeta_{k-1} \le \zeta < \zeta_k \\ (\zeta_{k+3}-\zeta)^5 - 6(\zeta_{k+2}-\zeta)^5 + 15(\zeta_{k+1}-\zeta)^5, & \zeta_k \le \zeta < \zeta_{k+1} \\ (\zeta_{k+3}-\zeta)^5 - 6(\zeta_{k+2}-\zeta)^5, & \zeta_{k+1} \le \zeta < \zeta_{k+2} \\ (\zeta_{k+3}-\zeta)^5, & \zeta_{k+2} \le \zeta < \zeta_{k+3} \\ 0, & \text{otherwise} \end{cases} \tag{2.1}$$

Here we give some of the main properties of quintic B-spline basis

- **Smoothness and continuity:** the Quintic B-spline and their derivatives up to the fourth derivative are continuous i.e. $\phi_k(\zeta) \in C^4[a,b]$
- **Finite support:** B-splines of order k have a finite support Supp $\phi_k(\zeta) = [\zeta_k, \zeta_{k+5}]$ $i.e.$ each quintic B-spline function $\phi_k(\zeta)$ extended over five points $\zeta_{k-2}, \zeta_{k-1}, \zeta_k, \zeta_{k+1}, \zeta_{k+2}$.
- **Positivity:** quintic B-spline functions $\phi_k(\zeta) > 0$ for $\zeta \in [\zeta_k, \zeta_{k+5}]$. Further properties and derivation of the quantic b-spline can be found in [34-36].

The nodal values of $\phi_i(\zeta)$ and its derivatives $\phi_i'(\zeta), \phi_i''(\zeta), \phi_i'''(\zeta), \phi_i^{(iv)}(\zeta)$ at these points are given in table (1).

2.2. Modified Quintic B-Spline.

Because of the extra knot points $\zeta_{i-2}, \zeta_{i-1}, \zeta_{i+1}, \zeta_{i+2}$ the collocation matrix of the quintic b-spline has the size $(N+5)\times(N+5)$. By using the boundary conditions, one can eliminate the extra points and the new basis functions are called the modified quintic B-spline. In the next parts of this paper, for simplicity the modified quintic B-spline will be called the MQBS method. MQBS can be derived from the classical quintic B-spline as follows [37]:

$$\begin{aligned}
\tilde{\phi}_1(\zeta) &= \phi_1(\zeta) + 2\phi_0(\zeta) + 3\phi_{-1}(\zeta),\\
\tilde{\phi}_2(\zeta) &= \phi_2(\zeta) - \phi_0(\zeta) - 2\phi_{-1}(\zeta),\\
\tilde{\phi}_j(\zeta) &= \phi_j(\zeta), \qquad for\ , j = 3,4,\dots,N-2,\\
\tilde{\phi}_{N-1}(\zeta) &= \phi_{N-1}(\zeta) - \phi_{N+1}(\zeta) - 2\phi_{N+2}(\zeta),\\
\tilde{\phi}_N(\zeta) &= \phi_N(\zeta) + 2\phi_{N+1}(\zeta) + 3\phi_{N+2}(\zeta), \qquad (2.2)
\end{aligned}$$

The nodal values for the modified b-spline function and their derivatives up to the Fourth derivative can be derived easily from equations (2.2).

The collocation method with MQBS is an efficient method solving the boundary Value problems [38]. Here we apply the modified bi-quintic B-spline function to find the approximate solution of the 2D biharmonic problem (1.1-1.2).

2.3. Collocation Method Using Modified Bi-Quintic B-Spline for Solving Biharmonic Equation.

Definition 2.1. Tensor product [39].

For any two matrices $\mathbf{G} = [g_{ij}] \in \mathbb{R}^{m\times n}$ and $\boldsymbol{H} = [h_{ij}] \in \mathbb{R}^{r\times s}$.The tensor product $G \otimes H = [g_{ij}H] \in \mathbb{R}^{mr\times ns}$ $i.e.$

$$G \otimes H \begin{bmatrix} g_{11}\mathrm{H} & g_{12}\mathrm{H} & \dots & g_{1n}\mathrm{H} \\ g_{21}\mathrm{H} & g_{22}\mathrm{H} & \dots & g_{2n}\mathrm{H} \\ \vdots & \vdots & & \vdots \\ g_{m1}\mathrm{H} & g_{m2}\mathrm{H} & \dots & g_{mn}\mathrm{H} \end{bmatrix}$$

Define the solution domain Ω is the triangle $a \le x \le b$, $c \le y \le d$, and let the boundary conditions (1.2) is described in detail as follows:

$$\begin{aligned}
&\psi(\mathrm{a,y}) = f_1(y), && \psi(\mathrm{b,y}) = f_2(y),\\
&\psi(\mathrm{x,c}) = f_3(x), && \psi(\mathrm{x,d}) = f_4(x), \qquad (2,3)\\
&\psi_x(\mathrm{a,y}) = g_1(y), && \psi_x(\mathrm{b,y}) = g_2(y),\\
&\psi_y(\mathrm{x,c}) = g_3(x), && \psi_y(\mathrm{x,d}) = g_4(x),
\end{aligned}$$

By dividing the $x-$interval into a uniform $N+1$ collocation points $x_i = a = x_0 \le x_1 \le \dots \le x_N = b$, $h_x = \frac{b-a}{N}$and the y− interval into a uniform $\mathrm{M}+1$ collocation points $y_k = c = y_0 \le y_1 \le \dots \le y_M = d$,with step size $h_y = \frac{d-c}{N}$.

Assume that the approximate solution of (1.1) is expressed as follows:

$$\tilde{\psi}(x,y) = \sum_{i=0}^{N}\sum_{j=0}^{M} \delta_{ij}\, \tilde{\phi}_{ij}(x,y) \qquad (2.4)$$

Where

$$\tilde{\phi}_{ij}(x,y) = \tilde{\phi}_i(x)\tilde{\phi}_j(y),$$

is the two-dimensional modified quintic B-spline basis functions. By substituting the approximate solution (2.4) into (1.1-1.2), the next theorem summarizes the solution's process.

***Theorem* 2.1.** If the function $\tilde{\psi}(x,y)$(2.4) is the approximate solution of the given biharmonic equation (1.1-1.2), then the unknown coefficients are determined by solving the linear system of equations.

$$[W;\mu]. \qquad (2.5)$$

Proof: from the series expansion of the approximate solution (2.4), one can deduce easily the first four derivatives according to the two variables x , y as follows

$$\begin{aligned}
\tilde{\psi}_x(x,y) &= \sum_{i=0}^{N}\sum_{j=0}^{M} \delta_{ij}\, \tilde{\phi}_i'(x)\tilde{\phi}_j(y),\\
\tilde{\psi}_{xx}(x,y) &= \sum_{i=0}^{N}\sum_{j=0}^{M} \delta_{ij}\tilde{\phi}_i''(x)\tilde{\phi}_j(y),\\
\tilde{\psi}_{xxx}(x,y) &= \sum_{i=0}^{N}\sum_{j=0}^{M} \delta_{ij}\, \tilde{\phi}_i'''(x)\tilde{\phi}_j(y), \qquad (2.6)\\
\tilde{\psi}_{xxxx}(x,y) &= \sum_{i=0}^{N}\sum_{j=0}^{M} \delta_{ij}\, \tilde{\phi}_i^{(4)}(x)\tilde{\phi}_j(y).
\end{aligned}$$

Similarly, the y derivatives can be deduced by the same way. And the fourth derivative term

$$\frac{\partial^4\psi(x,y)}{\partial x^2 \partial y^2} = \frac{\partial^2}{\partial x^2}\frac{\partial^2\psi(x,y)}{\partial y^2} = \frac{\partial^2}{\partial y^2}\frac{\partial^2\psi(x,y)}{\partial x^2} = \sum_{i=0}^{N}\sum_{j=0}^{M} \delta_{ij}\ \tilde{\phi}_i''(x)\tilde{\phi}_j''(y) \qquad (2.7)$$

by substituting the expansion (2.6-2.7) into the biharmonic equation (1.1), the following formula is obtained

$$\sum_{i=0}^{N}\sum_{j=0}^{M}\delta_{ij}\,\tilde{\phi}_i^{(4)}(x)\tilde{\phi}_j(y)+2\sum_{i=0}^{N}\sum_{j=0}^{M}\delta_{ij}\,\tilde{\phi}_i''(x)\tilde{\phi}_j''(y)+\sum_{i=0}^{N}\sum_{j=0}^{M}\delta_{ij}\tilde{\phi}_i(x)\tilde{\phi}_i^{(4)}(y)=\eta(x,y),\qquad (x,y)\in\Omega\,. \tag{2.8}$$

By inserting the collocation points $\{x_r\}_{r=0}^{N}$ and $\{y_s\}_{s=0}^{M}$ into equation (2.8) and representing the double summation with the tensor product notation, we obtain the following system of equations:

$$W\delta=\mu\,, \tag{2.9}$$

where
$W_{(N+1)^2\times(M+1)^2}=[v]_{(N+1)^2\times(M+1)^2}+2[k]_{(N+1)^2\times(M+1)^2}+[\vartheta]_{(N+1)^2\times(M+1)^2}$,

and each matrix is the tensor product of two $(N+1)\times(M+1)$ matrices as follows:

$v=v_1\otimes v_1\,,\quad v=k_1\otimes k_2\,,\quad \vartheta=\vartheta_1\otimes\vartheta_2.$
Where

$$v_1=\frac{1}{h_x^4}\begin{bmatrix}120&-240&120&0&0&0&\dots&0\\-240&600&-480&120&0&\dots&\dots&0\\120&-480&720&-480&120&0&\dots&0\\0&120&-480&720&-480&120&\vdots&0\\0&0&\vdots&\vdots&\vdots&\vdots&\vdots&0\\0&\dots&0&0&-480&720&-480&120\\0&\dots&\dots&0&120&-480&600&-240\\0&0&0&0&0&120&-240&120\end{bmatrix}_{(N+1)\times(N+1)}$$

$$v_2=\begin{bmatrix}121&-2&1&0&0&\dots&\dots&0\\28&65&26&1&0&\dots&\dots&0\\1&26&66&26&1&0&\dots&0\\0&1&26&66&26&1&\vdots&0\\0&0&\vdots&\vdots&\vdots&\vdots&\vdots&0\\0&\dots&0&1&26&66&26&1\\0&\dots&\dots&0&1&26&65&28\\0&\dots&\dots&\dots&0&1&-2&121\end{bmatrix}_{(M+1)\times(M+1)}$$

$$\vartheta_1=\frac{1}{h_y^4}\begin{bmatrix}120&-240&120&0&0&\dots&\dots&0\\-240&600&-480&120&0&\dots&\dots&0\\120&-480&720&-480&120&0&\dots&0\\0&120&-480&720&-480&120&\vdots&0\\0&0&\vdots&\vdots&\vdots&\vdots&\vdots&0\\0&\dots&0&0&-480&720&-480&120\\0&\dots&\dots&0&120&-480&600&-240\\0&\dots&\dots&\dots&0&120&-240&120\end{bmatrix}_{(M+1)\times(M+1)}$$

$$\vartheta_2=\begin{bmatrix}121&-2&1&0&0&\dots&\dots&0\\28&65&26&1&0&\dots&\dots&0\\1&26&66&26&1&0&\dots&0\\0&1&26&66&26&1&\vdots&0\\0&0&\vdots&\vdots&\vdots&\vdots&\vdots&0\\0&\dots&0&1&26&66&26&1\\0&\dots&\dots&0&1&26&65&28\\0&\dots&\dots&\dots&0&1&-2&121\end{bmatrix}_{(N+1)\times(N+1)}$$

$$\kappa_1=\frac{1}{h_x^2}\begin{bmatrix}20&-40&20&0&0&\dots&\dots&0\\80&-140&40&20&0&\dots&\dots&0\\20&40&-120&40&20&0&\dots&0\\0&20&40&-120&40&20&\vdots&0\\0&0&\vdots&\vdots&\vdots&\vdots&\vdots&0\\0&\dots&0&20&40&-120&40&20\\0&\dots&\dots&0&20&40&-140&80\\0&\dots&\dots&\dots&0&20&-40&20\end{bmatrix}_{(N+1)\times(N+1)}$$

and

$$\kappa_2 = \frac{1}{h_y^2}\begin{bmatrix} 20 & -40 & 20 & 0 & 0 & \dots & \dots & 0 \\ 80 & -140 & 40 & 20 & 0 & \dots & \dots & 0 \\ 20 & 40 & -120 & 40 & 20 & 0 & \dots & 0 \\ 0 & 20 & 40 & -120 & 40 & 20 & \vdots & 0 \\ 0 & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & 0 \\ 0 & \dots & 0 & 20 & 40 & -120 & 40 & 20 \\ 0 & \dots & \dots & 0 & 20 & 40 & -140 & 80 \\ 0 & \dots & \dots & \dots & 0 & 20 & -40 & 20 \end{bmatrix}_{(M+1)\times(M+1)}$$

$$\delta = [\delta_{00}, \delta_{01}, \dots, \delta_{0M}, \delta_{10}, \dots, \delta_{1M}, \dots, \delta_{NM}]^T$$

$$\mu = \left[\eta_{x_0,y_0}, \eta_{x_0,y_1}, \dots, \eta_{x_0,y_M}, \eta_{x_1,y_0}, \eta_{x_1,y_M}, \dots, \eta_{x_N,y_M}\right]^T$$

Where the dimension of above vectors δ and μ is $(N+1)(M+1)\times 1$. The boundary conditions are written in the form

$$\sum_{i=0}^{N}\sum_{j=0}^{M} \delta_{ij}\, \tilde{\phi}_i'(a)\tilde{\phi}_j'(y_k) = f_1(y_k), \qquad k = 0,1,\dots,M,$$

$$\sum_{i=0}^{N}\sum_{j=0}^{M} \delta_{ij}\, \tilde{\phi}_i'(b)\tilde{\phi}_j'(y_k) = f_2(y_k), \qquad k = 0,1,\dots,M,$$

$$\sum_{i=0}^{N}\sum_{j=0}^{M} \delta_{ij}\, \tilde{\phi}_i'(a)\tilde{\phi}_j'(y_k) = g_1(y_k), \qquad k = 0,1,\dots,M,$$

$$\sum_{i=0}^{N}\sum_{j=0}^{M} \delta_{ij}\, \tilde{\phi}_i'(b)\tilde{\phi}_j'(y_k) = g_2(y_k), \qquad k = 0,1,\dots,M,$$

$$\sum_{i=0}^{N}\sum_{j=0}^{M} \delta_{ij}\, \tilde{\phi}_i'(x_k)\tilde{\phi}_j'(c) = f_3(x_k), \qquad k = 0,1,\dots,N,$$

$$\sum_{i=0}^{N}\sum_{j=0}^{M} \delta_{ij}\, \tilde{\phi}_i'(x_k)\tilde{\phi}_j'(d) = f_4(x_k), \qquad k = 0,1,\dots,N,$$

$$\sum_{i=0}^{N}\sum_{j=0}^{M} \delta_{ij}\, \tilde{\phi}_i'(x_k)\tilde{\phi}_j'(c) = g_3(x_k), \qquad k = 0,1,\dots,N,$$

$$\sum_{i=0}^{N}\sum_{j=0}^{M} \delta_{ij}\, \tilde{\phi}_i'(x_k)\bar{\phi}_j'(d) = g_4(x_k), \qquad k = 0,1,\dots,N,$$

(2.10)

To enforce the approximate solution to achieve the boundary conditions, the 8 equations (2.3) are converted to matrix form, and they must be inserted into the main matrix of the system of linear equations (2.9) by replacing the rows corresponding to the boundary points x_0, x_1, y_0, y_1 by a specific order. The corresponding values in the RHS vector μ are replaced in the same way. For simplicity, consider $M = N$ so that the resulting linear system of equation is square $(N+1)^2 \times (N+1)^2$. The main matrix of such a system is nonsingular [38]. And the system is solved using the Gauss Jordan method. To obtain the approximate solution, the resulting coefficients were substituted in (2.4).

2.4. Error Estimate of Quintic B-Spline Collocation.

Theorem 2.2. Let $\zeta \in \Omega = [0,1], \phi(\zeta) \in C^{\gamma}[0,1], \gamma \geq 10 \in \mathbb{N}$. Define $\zeta_i = ih, i = 0,1,\dots,N, \zeta_0 = 0, \zeta_N = 1, h = \frac{1}{N}$. Assume $S(\zeta_i) = \vartheta(\zeta_i)$ and can be approximated by quartic B-spline as follows:

$$\mathcal{S}(\zeta i) = \sum_{k=-3}^{N+1} \delta_k\ \phi_k(\zeta_i), i = 0,1,\dots,N.$$

Then

$$\left|\mathcal{S}'(\zeta_i) - \vartheta'^{(\zeta_i)}\right| \approx \mathcal{O}(h^6),$$

$$\left|\mathcal{S}''^{(\zeta_i)} - \vartheta''^{(\zeta_i)}\right| \approx \mathcal{O}(h^4),$$

$$\left|\mathcal{S}'''^{(\zeta_i)} - \vartheta'''^{(\zeta_i)}\right| \approx \mathcal{O}(h^4),$$

$$\left|\mathcal{S}^{(4)}(\zeta_i) - \vartheta^{(4)}(\zeta_i)\right| \approx \mathcal{O}(h^2),$$

Proof: As $\mathcal{S}(\zeta_i) = \vartheta(\zeta_i)$, $\zeta_i \in \mathbf{\Omega}$, then $\mathcal{S}(\zeta_i)$ can be approximated by using the values in table (1) as follows:

$$\mathcal{S}_i = \beta_{i-2} + 26\beta_{i-1} + 66\beta_i + 26\beta_{i+1} + \beta_{i22},$$

$$h\mathcal{S}_i' = 5(\beta_{i-2} + 10\beta_{i-1} - 10\beta_{i+1} - \beta_{i+2}),$$

$$h^2\mathcal{S}_i'' = 20(\beta_{i-2} + 2\beta_{i-1} - 6\beta_i + 2\beta_{i+1} + \beta_{i+2}),$$

$$h^3\mathcal{S}_i''' = 60(\beta_{i-2} - 2\beta_{i-1} + 2\beta_{i+1} - \beta_{i+2}),$$

$$h^4\mathcal{S}_i^{(4)} = 120(\beta_{i-2} - 4\beta_{i-1} + 6\beta_i - 4\beta_{i+1} + \beta_{i+2}).$$

Define the invertible operator L with the notations:

$$\Lambda(\mathcal{S}(\zeta_i)) = \mathcal{S}(\zeta_{i+1}),$$

$$\Lambda(\mathcal{S}'(\zeta_i)) = \mathcal{S}'(\zeta_{i+1}),$$

$$\Lambda(\mathcal{S}''(\zeta_i)) = \mathcal{S}''(\zeta_{i+1}),$$

$$\Lambda(\mathcal{S}'''(\zeta_i)) = \mathcal{S}'''(\zeta_{i+1}),$$

$$\Lambda\left(\mathcal{S}^{(4)}(\zeta_i)\right) = \mathcal{S}^{(4)}(\zeta_{i+1}),$$

and the following relations can be hold for the first three derivatives

$$[\Lambda^{-2} + 26\Lambda^{-1} + 66 + 26\Lambda + \Lambda^2]\mathcal{S}'(\zeta_i) = \frac{5}{h}[\Lambda^{-2} + 10\Lambda^{-1} - 10\Lambda - \Lambda^2]\phi(\zeta_i),$$

$$[\Lambda^{-2} + 26\Lambda^{-1} + 66 + 26\Lambda + \Lambda^2]\mathcal{S}''(\zeta_i) = \frac{20}{h^2}[\Lambda^{-2} + 2\Lambda^{-1} - 6 + 2\Lambda + \Lambda^2]\phi(\zeta_i),$$

$$[\Lambda^{-2} + 26\Lambda^{-1} + 66 + 26\Lambda + \Lambda^2]\mathcal{S}'''(\zeta_i) = \frac{60}{h^3}[\Lambda^{-2} - 2\Lambda^{-1} + 2\Lambda - \Lambda^2]\phi(\zeta_i),$$

$$[\Lambda^{-2} + 26\Lambda^{-1} + 66 + 26\Lambda + \Lambda^2]\mathcal{S}^{(4)}(\zeta_i) = \frac{120}{h^4}[\Lambda^{-2} - 4\Lambda^{-1} + 6 - 4\Lambda + \Lambda^2]\phi(\zeta_i).$$

Where $\Lambda = e^{h\lambda}$ with $\lambda = \frac{d}{d\zeta}$ then [38]:

$\mathcal{S}'(\zeta_i) = \frac{5}{h}[e^{-2h\lambda} + 10e^{-h\lambda} - 10e^{h\lambda} - e^{2h\lambda}][e^{-2h\lambda} + 26e^{-h\lambda} + 66 + 26e^{h\lambda} + e^{2h\lambda}]^{-1}\phi(\zeta_i)$,
$\mathcal{S}''(\zeta_i) = \frac{20}{h^2}[e^{-2h\lambda} + 2e^{-h\lambda} - 6 + 2e^{h\lambda} + e^{2h\lambda}][e^{-2h\lambda} + 26e^{-h\lambda} + 66 + 26e^{h\lambda} + e^{2h\lambda}]^{-1}\phi(\zeta_i)$,
$\mathcal{S}'''(\zeta_i) = \frac{60}{h^3}[e^{-2h\lambda} - 2e^{-h\lambda} + 2e^{h\lambda} - e^{2h\lambda}][e^{-2h\lambda} + 26e^{-h\lambda} + 66 + 26e^{h\lambda} + e^{2h\lambda}]^{-1}\phi(\zeta_i)$,
$\mathcal{S}^{(4)}(\zeta_i) = \frac{120}{h^4}[e^{-2h\lambda} - 4e^{-h\lambda} + 6 - 4e^{h\lambda} + e^{2h\lambda}][e^{-2h\lambda} + 26e^{-h\lambda} + 66 + 26e^{h\lambda} + e^{2h\lambda}]^{-1}\phi(\zeta_i)$.

After expansion of the operator notation by means of power series, we can easily obtain the following

$$\mathcal{S}'(\zeta_i) = \left(\lambda + \frac{h^6\lambda^7}{5040} - \frac{h^8\lambda^9}{21600} + \cdots\right)\phi(\zeta_i)\ ,$$
$$\mathcal{S}''(\zeta_i) = \left(\lambda^2 + \frac{h^4\lambda^6}{720} - \frac{h^6\lambda^8}{3360} + \cdots\right)\phi(\zeta_i)\ ,$$
$$\mathcal{S}'''(\zeta_i) = \left(\lambda^3 - \frac{h^4\lambda^7}{240} + \frac{11h^6\lambda^9}{30240} + \cdots\right)\phi(\zeta_i)\ ,$$
$$\mathcal{S}^{(4)}(\zeta_i) = \left(\lambda^4 - \frac{h^2\lambda^6}{12} + \frac{h^4\lambda^8}{240} + \cdots\right)\phi(\zeta_i)\ ,$$

By applying the successive terms of differential operators λ and simplifying this yields the results and completes the proof.

III.NUMERICAL EXPERIMENTS

To test the efficiency of the proposed numerical scheme, several examples were proposed. The error is measured by using two different norms $L_\infty -$ norm , $L_2 -$ norm of error and relative L_2 −norm of error which are defined as follows.

$$L_\infty - error\ = \max_{\substack{0\le i\le N\\ 0\le j\le M}} \left|\psi_{\text{exact}}(x_i, y_j) - \psi_{\text{approx}}(x_i, y_j)\right|$$

$$L_2 - error = \sqrt{\frac{1}{NM}\sum_{i=0}^{N}\sum_{j=0}^{M}[\psi_{\text{exact}}(x_i, y_j) - \psi_{\text{approx}}(x_i, y_j)]^2}$$

$$\text{Relative } L_2 - \text{error} = \sqrt{\frac{\sum_{i=0}^{N}\sum_{j=0}^{M}[\psi_{\text{exact}}(x_i, y_j) - \psi_{\text{approx}}(x_i, y_j)]^2}{\sum_{j=0}^{M}[\psi_{\text{exact}}(x_i, y_j)]^2}}$$

***Problem** 1:* Consider the biharmonic equation
$\nabla^4\psi = f(x, y), 0 \le x \le 1, 0 \le y \le 1$,
where the solution for this problem is given in the exact form in [15] $\psi(x, y) = \frac{1}{64\pi^4}\sin(2\pi x)\sin(2\pi y)$. The **RHS** function $f(x, y) = \sin(2\pi x)\sin(2\pi y)$ and the values of boundary conditions can be easily calculated. Applying the proposed numerical scheme and the approximate solution is obtained. The comparison between our method and the Haar-wavelet method in [15] is summarized in table (2). The comparison shows the efficiency of our method for solving the biharmonic equations. Figure (1) shows the exact, approximate solution obtained by applying the MQBS method and $L_\infty -$ norm of error for the problem (1). For

TABLE 2.
THE COMPARISON OF L_∞– NORM OF ERROR BETWEEN THE MQBS AND HAAR WAVELET METHOD FOR PROBLEM (1).

Method		**L_∞– norm of error**
MQBS	N=11, M=11	$2.1672E(-06)$
	N=31, M=31	$2.6563E(-07)$
	N=51, M=51	$9.9503E(-08)$
Haar wavelet method [15]	*J=3*	1.7833E(–04)
	J=4	1.6766E(–04)
	J=5	1.6649E(–04)
	J=6	1.6590E(–04)

TABLE 3.
COMPARISON BETWEEN THE RELATIVE L_2 – ERROR FOR MQBS AND 1-D RBF COLLOCATION METHOD [40] FOR PROBLEM 1-A.

No. of points	**NSCM1 [40]**	**MQBS**
N = 11, M = 11	4.5E(– 01)	1.19 E(–02)
N = 21; M = 21	2.8E(– 01)	3.5 E(–03)
N = 51; M = 51	2.5E(– 01)	6.16 E(–04)

TABLE 4.
THE RESULTS OF L_∞ − ERROR AND L_2 − ERROR FOR MQBS COLLOCATION METHOD FOR PROBLEM 2.

No. of points	**L_∞ − error**	**L_2 − error**
N = 10, M = 10	1.647 E(−02)	5.53 E(−03)
N = 40, M = 40	4.4033 E(−03)	1.558 E(−03)
N = 100, M = 100	1.7720 E(−03)	6.3704 E(−04)
N = 120, M = 120	1.4776 E(−03)	5.3212 E(−04)

For the sake of comparison, a similar problem (**problem 1-a**) is presented which is founded in [40] with the exact solution $\psi(x, y) = \sin(2\pi x)\sin 2\pi y$. The RHS function, $f(x, y) = 64\pi 4\sin(2\pi x)\sin(2\pi y)$ and the boundary values $\psi = 0$ along the boundaries, $\psi_x(0, y) = \psi_x(1, y) = 2\pi\sin(2\pi y)$, $\psi(x, 0) = \psi_y(x, 1) = 2\pi\sin(2\pi x)$. The relative L_2 – error for both MQBS and NSCM1 [40] are summarized in the table (3).

***Problem 2**:* Let the biharmonic equation proposed by Bloor et al. [41]. The boundary values can be derived from the closed form solution.
$\psi(x, y) = x\cos(x)\, e^y, (x, y) \in [0, 1] \times [0, 1]$.

which is derived directly by applying the separation of variables on the homogeneous biharmonic equation. The method of MQBS was applied to such a problem and gave a good approximation of the approximate solution for different numbers of grid points. The results for $L_\infty -$ error and $L_2 -$ error are in table (4).

Table (4) shows the obtained approximate results by applying the MQBS method for problem 2. Both the $L_\infty -$ error and the $L_2 -$ error appears in the tabulated results for different grid points. The exact and approximate solutions as well as the absolute error for collocation grid 50×50 are shown in figure (2). And two.

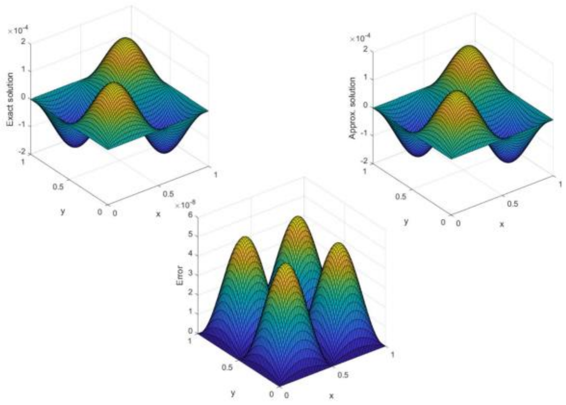

Figure 1. The figures of exact, approximate solution and $L_\infty -$ norm of error for problem (1).

TABLE 5.
THE RESULTS OF L_∞ − ERROR AND RELATIVE L_2 − ERROR FOR MQBS COLLOCATION METHOD FOR PROBLEM 3.

No. of points	L_∞ − error	RelativeL_2 − error
$N = 21, M = 21$	$1.649E-01$	$3.02E-02$
$N = 51, M = 51$	$7.02E-02$	$1.35\,E-03$
$N = 101, M = 101$	$3.58E-02$	$7.1\,E-03$

Types of error $L_\infty -$ error and $L_2 -$ error are drawn at various N, M and results are represented in figure (3).

Problem 3**:** Here we turned to another example for homogeneous biharmonic equation [42] is given by

$$\Delta^4\psi = 0, (x, y) \in [0,1] \times [0,1],$$

Where exact solution is

$$e^x \cos y + (x^2 + y^2)e^y \cos x.$$

Boundary conditions are derived easily from the exact solution. The MQBS solution is calculated and results are tabulated in the table (5). Also, the exact and approximate solutions are shown in figure (4).

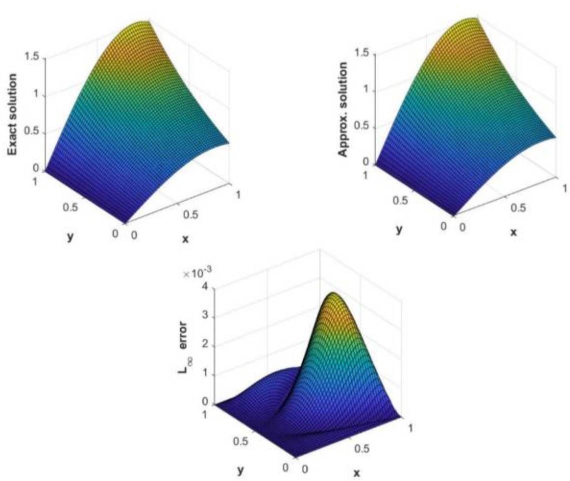

Figure 2. The exact and MQBS approximate solution for problem 2 with grid collocation points 50×50.

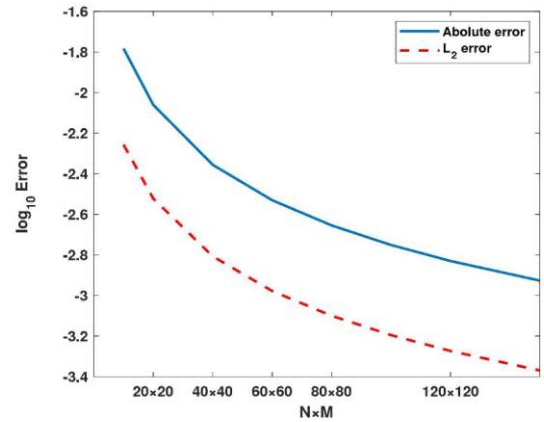

Figure 3. $log_{10}$of L_∞ and L_2 error formulas for problem 2 with various grid collocation points $N \times M$.

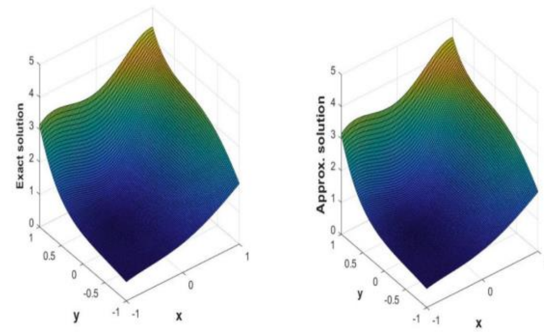

Figure 4. The exact and MQBS approximate solution for problem 3 with $N = 100, M = 100$.

IV. CONCLUSION

In the considered work, we introduce the modified bi-quintic B-spline (MQBS) scheme to approximate the solution of the non-homogeneous biharmonic problem. By applying this method, an accurate approximate solution was obtained. The proposed technique is simple and easy to implement and gives fast and accurate results for approximating the solution of the biharmonic problem. Three test problems were examined and comparisons with other numerical methods such as Haar wavelet and 1D- RBFs were tabulated. The comparisons ensure clearly that the MQBS is an effective and accurate tool among other numerical methods. The unsteady biharmonic equation will be studied in the future work and the numerical solution of such generalized model will be considered.

AUTHORS CONTRIBUTION

1. Conception or design of the work: ***Prof. Mohamed El-Gamel***
2. Data collection and tools: ***Dr. Atallah Elshenawy***
3. Data analysis and interpretation: ***Dr. Atallah Elshenawy***
4. Funding acquisition: No Funding
5. Investigation: ***Eng. Dina Reda***
6. Methodology: ***Eng. Dina Reda***
7. Project administration: ***Prof. Mohamed El-Gamel***

8. Software: ***Dr. Atallah Elshenawy& Eng. Dina Reda***
9. Supervision: ***Prof. Mohamed El-Gamel***
10. Drafting the article: ***Dr. Atallah Elshenawy & Eng. Dina Reda***
11. Critical revision of the article ***Dr. Atallah Elshenawy***
12. Final approval of the version to be published ***Prof. Mohamed El-Gamel & Dr. Atallah Elshenawy***

FUNDING STATEMEN:

The author did not receive any financial support of the research authorship and publication of this article.

DECLARATION OF CONFLICTING INTERESTS STATEMENT:

The author declared that there are no potential conflicts of interest with respect to the research authorship or publication of this article.

REFERENCES

[1.] A. Ali, K. Raslan, Variational iteration method for solving biharmonic equations, Phys. Lett. A. 370 (2007) 441–448.
[2.] A. Khalifa, The decomposition method for one dimensional biharmonic equations, Int. J. Simul. Model. 2 (2006) 59–64.
[3.] A. Hussain, S. T. Mohyud-Din, A. Yildirim, Solution of biharmonic equations using homotopy analysis method, Studies in Nonlinear Sciences 2 (1) (2011) 26–30.
[4.] L. Gao, Z. Zhang, Y. Xing, analytical solutions of biharmonic equation by the Fourier-Yang integral transform, Therm. Sci. 23 (2019) 91–91.
[5.] P. Bjrstad, fast numerical solution of the biharmonic Dirichlet problem on rectangles, SIAM J. Numer. Anal. 20 (1) (1983) 59–71.
[6.] L. W. Ehrlich, solving the biharmonic equation as coupled finite difference equations, SIAM J. Numer. Anal. 8 (2) (1971) 278–287.
[7.] A. Hamed, H. Hashim, M. Ahmed, M. B. S.K, Finite difference approximation method of bi harmonic equation in human face recognition, International Journal of Advances in Computer Science and Technology 2 (2013) 111–114.
[8.] X. liang Cheng, W. Han, H. ci Huang, some mixed finite element methods for biharmonic equation, J. Comput. Appl. Math 126 (1-2) (2000) 91–109.
[9.] F. Brezzi, M. Fortin, Mixed and Hybrid Finite Element Methods, Springer, Berlin, 1991.
[10.] R. Scholtz, A mixed method for fourth-order problems using the linear finite elements, RAIAO Numer. Anal. 15 (1978) 85–90.
[11.] H. Guo, Z. Zhang, Q. Zou, A c0linear finite element method for biharmonic problems, J. Sci. Comput. 74 (2018) 1397–1422.
[12.] Q. Zhuang, L. Chen, Legendre-Galerkin spectral-element method for the biharmonic equations and its applications, Comput. Math. with Appl. 74 (12) (2017) 2958–2968.
[13.] J. Shen, Efficient spectral-Galerkin method i. direct solvers of second-and fourth-order equations using Legendre polynomials, SIAM J. Sci. Comput. 15 (1994) 1489–1505.
[14.] S. Wenting, W. Xionghua, S. Chen, Chebyshev tau meshless method based on the integration differentiation for biharmonic-type equations on irregular domain, Eng. Anal. Bound. Elem. 36 (2012) 1787–1798.
[15.] Z. Shi, Y. yan Cao, Q. jiang Chen, Solving 2d and 3d Poisson equations and biharmonic equations by the Haar wavelet method, Appl. Math. Model. 36 (11) (2012) 5143–5161.
[16.] Z. Li, M. Lee, J. Chiang, Y. Liu, The Trefftz method using fundamental solutions for biharmonic equations, J. Comput. Appl. Math. 235 (2011) 4350–4367.
[17.] A. A. E.-M. M. El-Gamel, A. Mohsen, Sinc-Galerkin method for solving biharmonic problems, Appl. Math. Comput. 247 (2014) 386–396.
[18.] L. Boudjaj, A. Naji, F. Ghafrani, solving biharmonic equation as an optimal control problem using localized radial basis functions collocation method, Eng. Anal. Bound. Elem. 107 (2019) 208–217.
[19.] J. Chang, Q. Yang, C. Liu, B-spline method for solving boundary value problems of linear ordinary differential equations, Vol. 106, 2010, pp. 326–333.
[20.] N. Caglar, H. Caglar, B-spline method for solving linear system of second-order boundary value problems, Computers Mathematics with Applications 57 (5) (2009) 757–762.
[21.] A. Khalid, M.N. Naeem, P. Agarwal, A. Ghaffar, Z. Ullah , S. Jain ,Numerical approximation for the solution of linear sixth order boundary value problems by cubic b-spline, Adv. Differ. Equ. (2019) 492.
[22.] Y. Gupta, M. Kumar, B-spline method for solution of linear fourth order boundary value problem, Canadian Journal on Computing in Mathematics, Natural Sciences, Engineering and Medicine 2 (7) (2011) 166–169.
[23.] G. Loghmani, M. Ahmadinia, Numerical solution of sixth order boundary value problems with sixth degree b-spline functions, Applied Mathematics and Computation 186 (2) (2007) 992–999.
[24.] M. Kadalbajoo, P. Arora, B-spline collocation method for the singular-perturbation problem using artificial viscosity, Computers Mathematics with Applications 57 (4) (2009) 650–663.
[25.] J. Rashidinia, M. Ghasemi, B-spline collocation for solution of two-point boundary value problems, Journal of Computational and Applied Mathematics 235 (8) (2011) 2325–2342.
[26.] M. K. Kadalbajoo, V. Gupta, Numerical solution of singularly perturbed convection diffusion problem using parameter uniform b-spline collocation method, Journal of Mathematical Analysis and Applications 355 (1) (2009) 439–452.
[27.] I. Dag, A. Canivar, A. Sahin, Taylor- Galerkin and Taylor-collocation methods for the numerical solutions of Burgers' equation using b-splines, Communications in Nonlinear Science and Numerical Simulation 16 (7) (2011) 2696–2708.
[28.] R. Mittal, G. Arora, Quintic b-spline collocation method for numerical solution of the Kuramoto-Sivashinsky equation, Communications in Nonlinear Science and Numerical Simulation 15 (10) (2010) 2798–2808.
[29.] H. Caglar, M. Ozer, N. Caglar, the numerical solution of the one-dimensional heat equation by using third degree b-spline functions, Chaos, Solitons Fractals 38 (4) (2008) 1197–1201.
[30.] K. Raslan, Collocation method using quartic b-spline for the equal width (EW) equation, Applied Mathematics and Computation 168 (2) (2005) 795–805.
[31.] B. Saka, I. Dag, Quartic b-spline Galerkin approach to the numerical solution of the KDVB equation, Applied Mathematics and Computation 215 (2) (2009) 746–758.
[32.] R. Mohammadi, Sextic b-spline collocation method for solving Euler-Bernoulli beam models, Applied Mathematics and Computation 241 (2014) 151–166.
[33.] M. Tamsir, N. Dhiman, A. Chauhan, A. Chauhan, Solution of parabolic PDEs by modified quintic b-spline Crank-Nicolson collocation method, Ain Shams Eng. J. 12 (2) (2021) 2073– 2082.
[34.] C. de Boor, A Practical Guide to Splines, Springer Verlag, New York, 1978.
[35.] B. Saka, I.Dag, D. Irk, Quintic b-spline collocation method for numerical solution of the RLW equation, Anziam Journal 49 (3) (2008) 389–410.
[36.] M. Iqbal, M. Abbas, T. Nazir, Application of new quintic polynomial b-spline approximation for numerical investigation of kuramoto-sivashinsky equation, adv. differ. equ. 558 (2020) 1015–1018.
[37.] A. Bahan, Modification of quintic b-spline differential quadrature method to nonlinear Korteweg-de Vries equation and numerical experiments, Appl. Numer. Math. 167 (2021) 356– 374.
[38.] M. El-Gamel, A. El-Shenawy, The solution of a time-dependent problem by the b-spline method, J. Comput. Apl. Math. 267 (2014) 254–265.
[39.] H. Zhang, F. Ding, On the Kronecker products and their applications, J. Appl. Math. (2013) 8 pages.
[40.] N. Mai-Duy, R. Tanner, A collocation method based on one-dimensional rbf interpolation scheme for solving PDEs, Int. J. Numer. Method H. 17 (2) (2007) 165–186.
[41.] M. I. G Bloor, M. J Wilson, An approximate analytic solution method for the biharmonic problem, Proc. R. Soc. A: Math. Phys. Eng. Sci. 462 (2068) (2006) 1107–1121.
[42.] Z.-C. Li, M.-G. Lee, J. Y. Chiang, Y. P. Liu, The Trefftz method using fundamental solutions for biharmonic equations, J. Comput. Appl. Math. 235 (15) (2011) 4350–4367.

Title Arabic:

الحل العددي للمعادلة Biharmonic باستخدام طريقة التجميع المعتمدة على دوال b-spline الثنائية المعدلة من الرتبة الخامسة.

Arabic Abstract:

في العمل التالي تم استكشاف المعادلة Biharmonic. تم ايجاد الحل التقريبي عند النقط الواقعة في مجال الحل باستخدام طريقة التجميع على أساس دوال bspline الثنائية المعدلة من الرتبة الخامسة. دوال bspline الثنائية المعدلة تختفي عند نقاط الحدود. الخصائص الرئيسية لدوال bspline الثنائية المعدلة تم مناقشتها بالتفصيل. تعتمد الطريقة على اختزال المسألة المقترحة إلى مجموعه من المعادلات الخطية. يتم دمج الشروط الحديه في نظام المعادلات الخطية الناتجة بترتيب محدد لضمان تطابق الحل التقريبي مع الحل التام على نقاط الحدود. تم اضافة ثلاثة أمثلة عددية لضمان فعالية طريقة الحل، تم قياس دقة الحل التقريبي من خلال مقارنة ثلاثة أنواع مختلفة من الخطأ بين الحلول التقريبية والحلول التامة. تم توصيف النتائج بيانيا لتقييم أداء الطريقة المعروضة. الطريقة المقترحة سهلة التنفيذ، والنتائج العددية تضمن أن الطريقة تقرب حل مسألة Biharmonicبشكل جيد للغاية.